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Inverse Optimal Control Problem: the Sub-Riemannian Case ⋆

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Abstract: The object of this paper is to study the uniqueness of solutions of inverse control problems in the case where the dynamics is given by a control-affine system without drift and the costs are length and energy functionals.

Keywords: Optimal control, sub-Riemannian geometry, optimal trajectories, geodesics, inverse problem, nonholonomic systems, projective equivalence, affine equivalence.

1. INTRODUCTION

This paper is motivated by recent applications of optimal control theory to the study of human motions. Indeed, it is a widely accepted opinion in the neurophysiology community that human movements follow a decision that undergoes an optimality criterion (see Todorov (2006)). Finding this criterion amounts to solve what is called an *inverse optimal control problem*: given a set Γ of trajectories (obtained experimentally) and a class of optimal control problems – that is, a pair (control system, class $\mathcal{C}$ of costs) – suitable to model the system, identify a cost function φ in $\mathcal{C}$ such that the elements of Γ are minimizing trajectories of the optimal control problem associated with φ. Note that we restrict ourselves to integral costs, so the class $\mathcal{C}$ is actually the class of the infinitesimal costs.

The first two main aspects in the inverse optimal control problem are the question of existence of such an infinitesimal cost φ in the class $\mathcal{C}$, and the question of its uniqueness in this class. The existence part, even within the problems in classical Calculus of Variation, where $\mathcal{C}$ is the set of all smooth Lagrangians, is still an open problem, which attracted a lot of attention since the creation of Calculus of Variation (see a survey in Saunders (2010)). In the present paper the existence is assumed to hold a priori and the main question is the uniqueness of the cost φ in the class $\mathcal{C}$ or generically in the class $\mathcal{C}$, up to a multiplication by a positive constant.

It is easy to construct examples where the uniqueness does not hold. If the set Γ consists of unparameterized straight lines in $\mathbb{R}^2$, then in the class of length functionals with respect to Riemannian metrics on $\mathbb{R}^2$, there are functionals corresponding to Riemannian metrics with nonzero Gaussian curvature having Γ as their geodesics (see the example in subsection 3.1), so these functionals are not constantly proportional to the Euclidean length functional. Note also that by a classical theorem by Beltrami (1869), these functionals are the only ones with such property within this class. If one extends the class of functionals to Lagrangians, then one arrives to the variational version of Hilbert's fourth problem in dimension 2, which was solved by Hamel (1903), and provides a very rich class of Lagrangians having straight lines as extremals.

These examples are related to functionals without dynamical constraints, i.e. for which the space of admissible curves is defined by a trivial control system $\dot{x} = u$. If we consider the simplest class of optimal control problems, the linear-quadratic ones (the control system is linear and the cost is quadratic w.r.t. both state and control), the cost can be explicitly reconstructed from the optimal trajectories at least in the mono-input case, see Nori and Frezza (2004) and Berret and Jean (2016).

The present paper is devoted to the inverse problem for optimal control problems with a dynamical constraint given by a control-affine systems without drift and with two classes of functionals: the energy functionals (i.e. where the infinitesimal cost is quadratic with respect to control) and the length functionals (where the infinitesimal cost is just the square root of the infinitesimal energy cost). The first class of these optimal control problems (i.e. with the energy functionals) can be seen as a generalization of the class of linear-quadratic problems to the same extend as the energy functionals with respect to an arbitrary Riemannian metrics are generalizations of the corresponding Euclidean ones.

These two kinds of inverse problems can be reformulated in more geometric terms as problems of affine and projective equivalence of sub-Riemannian metrics, which in the case of Riemannian metrics are both classical: the classification of locally projectively equivalent Riemannian metrics under some natural regularity assumptions was done by Levi-Civita (1896) as an extension of the result of Dini (1870) for surfaces. The affinely equivalent Riemannian

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metrics are exactly the metrics with the same Levi-Civita connection and the description of the pairs of Riemannian metrics with this property can be attributed to Eisenhart (1923). The only complete classification of projectively equivalent metrics in a proper sub-Riemannian case was done far more recently in Zelenko (2006) for contact and quasi-contact sub-Riemannian metrics.

The paper is organized as follows. We first detail in section 2 the different notions of equivalence between infinitesimal costs and between metrics and show how they are related to the uniqueness of solutions of the corresponding inverse optimal control problems. We then expose in section 3 the results on equivalence of metrics in the Riemannian, contact and quasi-contact cases and their consequences for inverse problems. We adopt in this exposition the unifying point of view of the generalized Levi-Civita pairs and propose a general conjecture for the classification of affinely and projectively equivalent metrics. Finally we announce in section 4 our results on the equivalence of general sub-Riemannian metrics, showing in particular that for generic distributions all metrics are affinely rigid.

2. REDUCTION TO PROJECTIVE AND AFFINE EQUIVALENCE OF SUB-RIEMANNIAN METRICS

Let M be an n-dimensional smooth and connected manifold. Given a control system $\dot{q} = f(q,u)$ on M with a control space U we assign to any smooth infinitesimal cost $\varphi(q,u)$ the following family of optimal control problems parameterized by the initial and terminal times $a < b$ and by the initial and terminal points q_0, q_1:

$$\int_a^b \varphi(q,u)dt \to \min, \qquad (1)$$
$$\dot{q} = f(q,u),$$
$$q(a) = q_0, q(b) = q_1.$$

Note that, since the dynamical constraint and the cost are autonomous, by a time translation one can always make $a = 0$, but we prefer not to do it in order not to have unnecessary restrictions on possible time-parameterizations of minimimal trajectories.

Definition 1. We say that two infinitesimal costs φ and $\widetilde{\varphi}$ are equivalent via minimizers if the corresponding families of optimal control problems have the same minimizing trajectories.

It is clear that, in a given class $\mathcal{C}$, the existence of two distinct infinitesimal costs which are equivalent via minimizers implies that the inverse optimal control problem does not have uniqueness property in this class.

The set of minimizers is in general not easy to handle, it is easier to work with the extremals of (1). Recall that an *extremal trajectory* of (1) is a trajectory satisfying the conditions of the Pontryagin Maximum Principle, i.e. it is the projection q of a curve (q,p) on T^*M solution of some Hamiltonian equations arising from the maximisation w.r.t. u of $H(p,q,u,p^0) = \langle p, f(q,u)\rangle + p^0\varphi(q,u)$, where $p^0 \leq 0$ is a scalar. Every minimizer is an extremal trajectory. This suggests a second notion of equivalence.

Definition 2. We say that two infinitesimal costs φ and $\widetilde{\varphi}$ are equivalent via extremal trajectories if the corresponding families of optimal control problems have the same extremal trajectories.

Both notions of equivalence are different in general, but we will see below that in particular cases the first one implies the second.

We consider now a control-affine system without drift,

$$\dot{q} = \sum_{i=1}^{m} u_i X_i(q), \qquad q \in M, \qquad (2)$$

where $X_1, \dots, X_m$ are vector fields on M and the control $u = (u_1, \dots, u_m)$ takes values in $\mathbb{R}^m$. We assume that the Lie algebra generated by the vector fields $X_1, \dots, X_m$ is of full rank, i.e. $\dim \mathrm{Lie}(X_1, \dots, X_m)(q) = n$ for every $q \in M$, which guarantees that the system is controllable. Such a system is called a *Lie bracket generating nonholonomic system*. We make the additional assumption that $D(q) = \mathrm{span}\{X_1(q), \dots, X_m(q)\}$ is of constant rank equal to m, which implies that D defines a rank m distribution (i.e. a rank m subbundle of TM), $X_1, \dots, X_m$ being *a frame* of the distribution. Note that we can always make this assumption in a neighbourhood of a generic point (up to reducing m).

Define $\mathcal{C}$ as the set of smooth functions $g : M \times \mathbb{R}^m \to \mathbb{R}$, $(q,u) \mapsto g(q)(u)$, such that for every $q \in M$, $g(q)(\cdot)$ is a positive definite quadratic form. From a more geometric viewpoint, we can see g as a function on D and write $g(\dot{q})$ instead of $g(q)(u)$ for $\dot{q}$ satisfying (2). Thus the set $\mathcal{C}$ appears as the set of the sub-Riemannian metrics on (M,D) and, in the particular case where $m = n$ (and so $D = TM$), $\mathcal{C}$ is the set of the Riemannian metrics on M. Any $g \in \mathcal{C}$ is the infinitesimal cost for the *energy functional*, while $\sqrt{g}$ is the infinitesimal cost for the *length functional* associated with the sub-Riemannian metric g.

Since two constantly proportional metrics define the same energy and length minimizers, the problem of injectivity can be stated as follows.

Inverse sub-Riemannian problems *Let M be a manifold and D a distribution on M. Can we recover g in a unique way, up to a multiplicative constant, from the knowledge of all energy minimizers of (M,D,g)? And from the knowledge of all length minimizers of (M,D,g)?*

When the answer to one of the above questions is positive, we say that the corresponding inverse sub-Riemannian problem for (M,D) is injective. Obviously injectivity for the problem with length minimizers implies injectivity for the problem with energy minimizers.

Now let us try to characterize the injectivity of the above problem through equivalence via extremal trajectories. Given a sub-Riemannian metric g on (M,D), the extremal trajectories of the energy functional are called the *sub-Riemannian geodesics*. There are two type of sub-Riemannian geodesics, normal and abnormal (see Montgomery (2002) or Rifford (2014) for details). The alternative is not exclusive, a geodesic can be both normal and abnormal. If it is not the case we will say that the geodesic is either strictly normal or strictly abnormal. In

the Riemannian case (i.e. $D = TM$) there are no abnormal geodesics and the normal geodesics coincide with the usual geodesics. Note that we have an explicit description of the normal geodesics: they are the projections on M of the trajectories in T^*M of the Hamiltonian vector field $\boldsymbol{h_g}$, where the Hamiltonian $h_g : T^*M \to \mathbb{R}$ is defined as

$$h_g(q,p) = \frac{1}{2}\|p|_{D(q)}\|_g^2, \quad q \in M,\ p \in T_q^*M, \tag{3}$$

where

$$\|p|_{D(q)}\|_g := \max\left\{ \left\langle p, \sum_i u_i X_i(q) \right\rangle \ :\ g(q)(u) = 1 \right\}.$$

Note that h_g is quadratic on each fiber T_q^*M.

Two facts are worth to mention here.

Fact 1 Normal geodesics are locally energy minimizers.

Fact 2 Abnormal geodesics are characterized only by the distribution D, they do not depend on the metric g.

The last fact implies that for equivalence via extremals we only need to examine normal geodesics and even only strictly normal geodesics.

Besides, note that the length functional does not depend on the parameterization of the trajectory, so the set of minimizers and the set of geodesics of the length functional are invariant under an arbitrary time-reparameterization. On the contrary, it is classical in Riemannian and well known in sub-Riemannian geometries that a minimizer q_u of the energy functional is a minimizer of the length functional such that $g(q_u(t))(u(t))$ is constant, so the set of the minimizers of the energy functional is invariant only under affine time-reparameterizations. In other terms, $t \mapsto q_u(t)$ and $\tau \mapsto q_u(\psi(\tau))$ are simultaneously minimizers of the energy functional if and only if $\psi(\tau) = \alpha\tau + \beta$ for $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. The same holds true for normal geodesics, it follows from (3).

Definition 3. We say that two sub-Riemannian metrics on (M, D) are *projectively equivalent* [1] if they have the same normal geodesics as unparameterized curves. We say that they are *affinely equivalent* if they have the same normal geodesics as parameterized curves or equivalently, the same normal geodesics up to affine time-reparameterization.

Note that affine equivalence implies projective equivalence but in general the two notions do not coincide. For instance, on $M = \mathbb{R}$, all metrics are projectively equivalent to each other while two metrics are affinely equivalent if and only if they are constantly proportional.

Equivalences of metrics are related with the equivalence of infinitesimal costs given in Definition 2 as follows.

Lemma 4. Let g and $\widetilde{g}$ be two sub-Riemannian metrics on (M, D). Then,

- g and $\widetilde{g}$ are affinely equivalent if and only if g and $\widetilde{g}$ are equivalent via extremal trajectories;
- g and $\widetilde{g}$ are projectively equivalent if and only if $\sqrt{g}$ and $\sqrt{\widetilde{g}}$ are equivalent via extremal trajectories.

Proof. Let us prove the first point, the same argument holds for the second one. From Fact 2 above, it is clear that affine equivalence implies equivalence via extremals. Conversely, assume g and $\widetilde{g}$ are equivalent via extremal trajectories. From Fact 2 again, it implies that they have the same strictly normal geodesics. But it results from (Agrachev et al., 2016, Prop. 3.12 and 5.23) that generic normal geodesics are strictly normal (generic here means for an open and dense subset of the initial condition $p \in T_q^*M$ of the Hamiltonian equations). The latter fact, together with a characterization of affine equivalence through the Hamiltonian vector fields $\boldsymbol{h_g}$ and $\boldsymbol{h_{\tilde{g}}}$ (see Jean et al. (2016)), allows to prove that g and $\widetilde{g}$ have the same normal geodesics, and so that they are affinely equivalent.

The following result clarifies the relationship between the sub-Riemannian inverse problem and the projective and affine equivalences. It is based on Facts 1 and 2 on sub-Riemannian geodesics stated above.

Lemma 5. Let $g, \tilde{g}$ be sub-Riemannian metrics on (M, D).

- If g and $\tilde{g}$ are equivalent via minimizers, then they are equivalent via extremals, and so g and $\tilde{g}$ are affinely equivalent.
- If $\sqrt{g}$ and $\sqrt{\tilde{g}}$ are equivalent via minimizers, then they are equivalent via extremals, and so g and $\tilde{g}$ are projectively equivalent.

Proof. Assume that g and $\tilde{g}$ have the same energy minimizers. Let γ be a geodesic of g. Either it is an abnormal geodesics, and then by Fact 2 it is also an abnormal geodesic of $\tilde{g}$. Or it is a normal geodesic of g; in this case by Fact 1 every sufficiently short piece of γ is an energy minimizer for g, therefore γ is an energy minimizer and then a geodesic for $\tilde{g}$. In both cases γ is a geodesic of $\tilde{g}$. Thus g and $\widetilde{g}$ are equivalent via extremals and by Lemma 4 they are affinely equivalent. The same argument holds for projective equivalence.

Again the trivial example is the case of two constantly proportional metrics g and cg, $c > 0$. We thus say that these metrics are *trivially* (projectively or affinely) equivalent.

Definition 6. A metric g on (M, D) is said to be *projectively rigid* (resp. *affinely rigid*) if it admits no nontrivial projectively (resp. affinely) equivalent metric.

Corollary 7. If every metric on (M, D) is affinely (resp. projectively) rigid, then the inverse sub-Riemannian problem on (M, D) with energy (resp. length) minimizers is injective.

The consequence of this result is that we can replace the analysis of the inverse sub-Riemannian problem by the analysis of affine and projective equivalences. The latter is more tractable since it relies on the analysis of the Hamiltonian equations of the normal geodesics.

We finish the section with a discussion about the relationship between the introduced equivalence relations and conformal sub-Riemannian metrics, which will be very important in the sequel, because in many cases if two sub-Riemannian metrics are projectively equivalent then they must be conformal. Recall that a sub-Riemannian metric $\tilde{g}$ on (M, D) is said to be *conformal* to another sub-Riemannian metric g if $\tilde{g} = \alpha^2 g$, where $\alpha : M \to \mathbb{R}$ is a nonvanishing smooth function. The trivial case of con-

[1] Projective equivalence is sometimes called geodesic equivalence.

stantly proportional metrics is the particular case where α is constant.

Lemma 8. Let $g, \tilde{g}$ be two affinely equivalent metrics on (M, D) which are conformal to each other. Then they are constantly proportional.

Proof. Recall that $g((q_u(t))(u(t))$ is constant along any normal geodesic q_u of the metric g. Since g and $\tilde{g}$ have the same geodesics, $g((q_u(t))(u(t))$, $\tilde{g}((q_u(t))(u(t))$ and thus $\alpha((q_u(t))$, are constant along these geodesics. From (Rifford and Trélat, 2005, Th. 1.1), the normal geodesics issued from a point fill a dense subset of a neighbourhood of this point, therefore the function α is locally constant. The conclusion follows from the connectedness of M.

Note that two conformal metrics are not projectively equivalent in general. We actually conjecture that the latter situation occurs only when $\dim M = 1$ (or when the metrics are constantly proportional to each other).

Using Lemma 8, we obtain another relation between inverse sub-Riemannian problem and projective rigidity.

Corollary 9. If all projectively (or affinely) equivalent metrics on (M, D) are conformal to each other, then the inverse sub-Riemannian problem on (M, D) with energy minimizers is injective.

3. PROJECTIVE AND AFFINE EQUIVALENCE: EXAMPLES AND FIRST CASES

3.1 Examples of nontrivial equivalent metrics

Euclidean metrics. Let $M = \mathbb{R}^n$ and let (2) be the system defined by n constant linearly independent vector fields $f_1, \dots, f_n$ on $\mathbb{R}^n$ (i.e. $D = \mathbb{R}^n$). Any quadratic costs of the form $g(u) = u^T Q u$, where Q is a positive definite symmetric matrix, defines a Euclidean metric on $\mathbb{R}^n$ and the associated energy minimizers are straight lines parameterized with constant velocity. As a consequence, all Euclidean metrics are affinely equivalent.

Southern hemisphere. Consider in $\mathbb{R}^3$ the horizontal plane $P = \{z = 0\}$ and the southern hemisphere S of the unit sphere centered at $p = (0, 0, 1)$. Let g be a Euclidean metric on $P \simeq \mathbb{R}^2$ and $\tilde{g}$ the metric on S induced by the Euclidean metric of $\mathbb{R}^3$. The stereographic projection $\pi : S \to P$ from p (also called *gnomonic map*) sends the big circles on the hemisphere to the straight lines on the plane, see Fig. 1. As a consequence, the metrics g and $\pi_* \tilde{g}$ on P are projectively equivalent. However they are not constantly proportional since the Gaussian curvature of a plane is zero and the one of a sphere is not. And they are not affinely equivalent since the restriction of the metrics to the straight lines are conformal to each other and not constantly proportional (see (5) below).

Levi-Civita pairs. Fix positive integers $k_1, \dots, k_N$ such that $k_1 + \cdots + k_N = n$. Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_N)$, where $\bar{x}_s = (x_s^1, \dots, x_s^{k_s})$, be standard coordinates in $\mathbb{R}^n = \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_N}$. For any $s \in \{1, \dots, N\}$, let D_s be a Lie bracket generating distribution on $\mathbb{R}^{k_s}$. Consider the distribution D on $\mathbb{R}^n$ which is obtained as the product of the distributions D_s. Namely, if $\pi_s : \mathbb{R}^n \to \mathbb{R}^{k_s}$, $s = 1, \dots, N$, are the canonical projections, then

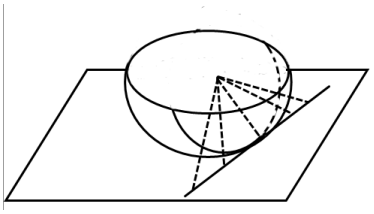


Fig. 1. Gnomonic map

$$D(\bar{x}) = \left\{ v \in T_{\bar{x}}\mathbb{R}^n : (\pi_s)_*(v) \in D_s\big(\pi_s(\bar{x})\big),\ s = 1, \dots, N \right\}.$$

We call such a distribution D a *product distribution with N factors.*

For every $s \in \{1, \dots, N\}$, choose a sub-Riemannian metric b_s on $(\mathbb{R}^{k_s}, D_s)$ and a function β_s depending only on the variables $\bar{x}_s$ such that β_s is constant if $k_s > 1$ and $\beta_s(0) \neq \beta_l(0)$ for $l \neq s$.

For $\dot{\bar{x}}$ in $D(\bar{x})$, set

$$\begin{cases} g(\dot{\bar{x}}) = \displaystyle\sum_{s=1}^{N} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s), \\ \tilde{g}(\dot{\bar{x}}) = \displaystyle\sum_{s=1}^{N} \lambda_s(\bar{x}) \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s). \end{cases} \tag{4}$$

where

$$\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=1}^{N} \beta_l(\bar{x}_l),\ \gamma_s(\bar{x}) = \prod_{l \neq s} \Big| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \Big|.$$

It can be shown that g and $\tilde{g}$ are projectively equivalent, and that they are also affinely equivalent if all functions β_s, $s = 1, \dots, N$ are constant (and so all functions λ_s and γ_s are constant as well).

Note that the first two examples of this subsection take the form above up to a change of coordinates:

- Euclidean metrics: let $g, \tilde{g}$ be Euclidean metrics associated with the positive definite symmetric matrices $Q, \tilde{Q}$ respectively. One can then choose a basis $f_1, \dots, f_n$ orthonormal w.r.t. Q that diagonalizes $\tilde{Q}$. Using these vectors to define the system (2), we have: $g(u) = u_1^2 + \cdots + u_n^2$ and $\tilde{g}(u) = \lambda_1 u_1^2 + \cdots + \lambda_n u_n^2$.
- Southern hemisphere: in polar coordinates on P identified with $\mathbb{R}^2$, we have $g = dr^2 + r^2 d\phi^2$ and
$$\pi_* \tilde{g} = \frac{1}{1 + r^2} \left(\frac{1}{1 + r^2} dr^2 + r^2 d\phi^2 \right). \tag{5}$$
In this example the functions β_s are not constant.

Using coordinates, we can also extend our construction in $\mathbb{R}^n$ to an n-dimensional manifold M.

Definition 10. We say that a pair $(g, \tilde{g})$ of sub-Riemannian metrics on (M, D) form a *generalized Levi-Civita pair* at a point $q \in M$, if there is a local coordinate system in a neighbourhood of q, in which D takes the form of a product distribution with $N \geq 2$ factors, and the metrics g and $\tilde{g}$ have the form (4). We say that such a pair has *constant coefficients* if the coordinate system can be chosen so that the functions β_s, $s = 1, \dots, N$ are constant.

Generalized Levi-Civita pairs are projectively equivalent, and they are affinely equivalent if they have constant coefficients.

This definition is inspired by the classification in the Riemannian case appearing in Levi-Civita (1896). Note however that, in the Riemannian case, the distribution $D = TM$ takes the form of a product in any system of coordinates, so that Levi-Civita pairs always exist locally.

3.2 Transition operator

Let $g, \tilde{g}$ be two sub-Riemannian metrics on (M, D). The *transition operator* from the metric g to the metric $\tilde{g}$ at a point $q \in M$ is the linear operator $S_q : D(q) \to D(q)$ s.t.

$$G(q)(v_1, v_2) = \tilde{G}(q)(S_q v_1, v_2), \quad v_1, v_2 \in D(q),$$

where $G, \tilde{G}$ are the bilinear forms defined by $g, \tilde{g}$ respectively. The operator S_q is positive definite and self-adjoint w.r.t. g. The number of distinct eigenvalues of S_q at a point q will be called the *spectral size* of S_q and is denoted by $N(q)$. A point $q_0 \in M$ is called *stable* w.r.t. the ordered pair $(g, \tilde{g})$ if the function $N(q)$ is constant in some neighbourhood of q_0. The set of points stable w.r.t. $(g, \tilde{g})$ is generic in M.

Note that the integer N appearing in the construction of generalized Levi-Civita pairs is the spectral size of the transition operator from g to $\tilde{g}$.

3.3 The Riemannian case

In the Riemannian case, that is when $D = TM$, the local classification of projectively equivalent metrics near generic points has been established by Dini (1870) in dimension 2, then by Levi-Civita (1896) in any dimension. The classification of affinely equivalent metrics is a consequence of (Eisenhart, 1923, Th. p. 303). We summarize all these results in the following theorem.

Theorem 11. Assume $\dim M > 1$. Then two Riemannian metrics on M are non trivially projectively equivalent in a neighbourhood of a stable point q if and only if they form a generalized Levi-Civita pair at q. They are moreover affinely equivalent if the pair has constant coefficients.

A consequence of Theorem 11 is that the inverse Riemannian problems are not injective in general. However it shows that the metrics that are not projectively rigid have a very specific form. In fact, Levi-Civita also shown that if g and $\tilde{g}$ have the form (4), the integer N being the spectral size of the transition operator, in addition to the kinetic energy integral, the geodesic flow of g admits $N-1$ first integrals which are quadratic with respect to velocities (all these N integrals are in involution). In particular, it admits the following integral:

$$\Big(\prod_{s=1}^{N} \lambda_s\Big)^{-\frac{2}{N+1}} \tilde{g}(u) \tag{6}$$

(see also Matveev and Topalov (2003), where this integral is attributed to Painlevé). Further, if $\tilde{g}$ is not constantly proportional to g, then the spectral size N is greater than 1. Hence the Painlevé integral (6) is not constantly proportional to the kinetic energy. In other words the geodesic flow of g admits a nontrivial integral which is quadratic with respect to the velocities. From this and Kruglikov and Matveev (2016) we obtain the following:

Corollary 12. Generic Riemannian metrics are projectively (and so affinely) rigid.

3.4 The case of corank one distributions

From the previous results arise the following questions.

Two main questions *Are the generalized Levi-Civita pairs the only pairs of locally projectively equivalent sub-Riemannian metrics under natural regularity assumptions? And are the pairs with constant coefficients the only pairs of locally affinely equivalent sub-Riemannian metrics?*

The answer yet is known to be positive beyond the Riemannian case only for sub-Riemannian metrics on contact and quasi-contact distributions, which are typical cases of corank 1 distributions (i.e. $m = n-1$). Recall that a *contact distribution* D on a $2k+1$ dimensional manifold M is a rank $2k$ distribution for which there exists a 1-form ω such that at every $q \in M$, $D(q) = \ker \omega(q)$ and $d\omega(q)|_{D(q)}$ is non-degenerate. A *quasi-contact distribution* D on a $2k$ dimensional manifold M is a rank $2k-1$ distribution for which there exists a 1-form ω such that at every $q \in M$, $D(q) = \ker \omega(q)$ and $d\omega(q)|_{D(q)}$ has a one-dimensional kernel.

Theorem 13. (Zelenko (2006)). Two sub-Riemannian metrics on a contact or a quasi-contact distribution are non trivially projectively equivalent in a neighbourhood of a stable point q if and only if they form a generalized Levi-Civita pair at q.

Since contact distributions are never locally equivalent to a product distribution, they do not admit generalized Levi-Civita pairs.

Corollary 14. On a contact distribution, every sub-Riemannian metric is projectively rigid.

For a generic corank one distribution D on an odd dimensional manifold M, there is an open and dense subset of M where D is locally contact. By continuity we obtain the following result.

Corollary 15. Let M be an odd-dimensional manifold. Then, for a generic corank one distribution on M, all metrics are projectively rigid.

Note that a quasi-contact distribution admits a product structure $\mathbb{R} \times \{\text{a contact distribution}\}$, hence by Theorem 13 the inverse sub-Riemannian problem is not injective in general on such distributions. However it is easy to see that generic metrics on quasi-contact distribution are projectively rigid.

4. CONFORMAL RIGIDITY AND GENERIC AFFINE RIGIDITY IN SUB-RIEMANNIAN CASE

The two main questions formulated in the previous section are widely open yet. In this section we want to announce our recent progress in this direction, and in particular results on affine rigidity and some results toward projective rigidity. For the details and proofs we refer to our forthcoming paper Jean et al. (2016).

Assume that the transition operator of a pair of sub-Riemannian metrics g and $\widetilde{g}$ has a spectral size N and eigenvalues $\lambda_1, \ldots, \lambda_N$ in a neighbourhood of a point q.

Proposition 16. If g and $\widetilde{g}$ are projectively equivalent, then the normal extremal flow of the metric g admits the Painlevé type integral

$$(\prod_{s=1}^{N} \lambda_s)^{-\frac{2}{N+1}} h_{\tilde{g}}, \tag{7}$$

where $h_{\tilde{g}}$ is the normal sub-Riemannian Hamiltonian for the metric $\tilde{g}$ defined by (3).

This result is important in view of our main questions above since, if $(g, \widetilde{g})$ is a generalized Levi-Civita pair, then the normal extremal flow of g admits a Painlevé integral (7) (actually it admits N integrals in involution, some of which may coincide up to a constant multiple, as in the Riemannian case).

Definition 17. We say that a sub-Riemannian metric g is *conformally rigid with respect to projective equivalence* if any projectively equivalent metric to it is also conformal to it.

Note that by Corollary 9 any metric which is conformally rigid with respect to projective equivalence is affinely rigid. Applying to sub-Riemannian metrics an argument similar to the one in Kruglikov and Matveev (2016), we obtain the following result.

Corollary 18. Given a distribution D on M, generic sub-Riemannian metrics on (M, D) are conformally rigid with respect to projective equivalence, and therefore are affinely rigid.

Now, let q_0 be a stable point of a pair $(g, \tilde{g})$, and $X_1, \dots, X_m$ be a local frame of D in a neighbourhood U of q_0 which consists of eigenvectors of the transition operator. Let $\lambda_1, \dots, \lambda_m$ be the corresponding eigenvalues (they can be repeated and, as a set, they coincide with the set of eigenvalues used in Proposition 16). Using results in Zelenko (2006), we obtain the following consequence of Proposition 16.

Proposition 19. If two sub-Riemannian metrics $g, \tilde{g}$ on (M, D) are projectively equivalent near a stable point q_0, then for any q in the neighbourhood U as above the following property holds for any $0 \leq i, j \leq m$:

$$[X_i, X_j](q) \notin D(q) \quad \Longrightarrow \quad \lambda_i(q) = \lambda_j(q).$$

If moreover $g, \tilde{g}$ are affinely equivalent near q_0, then all eigenvalues λ_i are constants.

Now given a frame $(X_1, \dots, X_m)$ of a distribution D, define the graph of the frame at q as follows: the vertices of the graph are elements of $\{1, \dots, m\}$ and vertex i is connected to vertex j if $[X_i, X_j](q) \notin D(q)$.

Corollary 20. If the graph of any frame of the distribution D is connected, then any sub-Riemannian metric g on (M, D) is affinely rigid and is conformally rigid with respect to projective equivalence.

Given a distribution D with local frame $(X_1, \dots X_m)$, set $D^2(q) := \operatorname{span}\{X_i(q), [X_i, X_j](q)\}_{1 \leq i,j \leq m}$.

Corollary 21. Assume that D is free up to the second step, i.e. D^2 is a distribution of rank $m(m+1)/2$. Then any sub-Riemannian metric g on (M, D) is affinely rigid and is conformally rigid with respect to projective equivalence.

Proof. Indeed, in this case the graph of any local frame of D is complete and Corollary 20 applies.

Corollary 22. Assume $\dim M \geq m(m+1)/2$. Then for generic rank m distributions on M, any sub-Riemannian metrics on them is affinely rigid and is conformally rigid with respect to projective equivalence.

Proof. In this case generic distributions are free up to the second step at generic points and we can use Corollary 21.

REFERENCES

Agrachev, A., Barilari, D., and Rizzi, L. (2016). *Curvature: a variational approach.* Memoirs of the AMS. In press.

Beltrami, E. (1869). Teoria fondamentale degli spazii di curvatura costante. *Annali di matem*, 2, 232–255. Ser.2.

Berret, B. and Jean, F. (2016). Why don't we move slower? the value of time in the neural control of action. *The Journal of Neuroscience*, 36 (4), 1056–1070.

Dini, U. (1870). Sopra un problema che si presenta nella teoria generale delle rappresentazione geografiche di una superficie su di un' altra. *Annali di matem*, 2, 269–293. Ser.3.

Eisenhart, L.P. (1923). Symmetric tensors of the second order whose first covariant derivatives are zero. *Trans. Amer. Math. Soc.*, 25(2), 297–306.

Hamel, G. (1903). Über die Geometrieen, in denen die Geraden die Kürzesten sind. *Math. Ann.*, 57(2), 231–264.

Jean, F., Maslovskaya, S., and Zelenko, I. (2016). On projective and affine rigidity of sub-Riemannian metrics. In preparation.

Kruglikov, B. and Matveev, V. (2016). The geodesic flow of a generic metric does not admit nontrivial integrals polynomial in momenta. *Nonlinearity*, 29, 1775–1768.

Levi-Civita, T. (1896). Sulle trasformazioni delle equazioni dinamiche. *Ann. Mat. Ser. 2a*, 24, 255–300.

Matveev, V.S. and Topalov, P. (2003). Geodesic equivalence via integrability. *Geometriae Dedicat*, 96, 91–115.

Montgomery, R. (2002). *A tour of sub-Riemannian geometries, their geodesics and applications.* Math. Surveys and Monographs. American Math. Soc.

Nori, F. and Frezza, R. (2004). Linear optimal control problems and quadratic cost functions estimation. In *12th Mediterranean Conference on Control and Automation, MED'04. Kusadasi, Aydin, Turkey.*

Rifford, L. (2014). *Sub-Riemannian Geometry and Optimal Transport.* SpringerBriefs in Mathematics. Springer International Publishing.

Rifford, L. and Trélat, E. (2005). Morse-Sard type results in sub-Riemannian geometry. *Math. Ann.*, 332(1), 145–159.

Saunders, D. (2010). Thirty years of the inverse problem in the calculus of variations. *Reports on Mathematical Physics*, 66(1), 43 – 53.

Todorov, E. (2006). *Bayesian Brain: Probabilistic Approaches to Neural Coding (Doya K. et al. Eds)*, chapter Optimal control theory, chapter 12, 269–298. MIT Press.

Zelenko, I. (2006). On the geodesic equivalence of Riemannian metrics and sub-Riemannian metrics on distributions of corank 1. *J. Math. Sci. (N. Y.)*, 135(4), 3168–3194.