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LEVEL-SET APPROACH FOR REACHABILITY ANALYSIS OF HYBRID SYSTEMS UNDER LAG CONSTRAINTS*

G. GRANATO† AND H. ZIDANI‡

Abstract. This study aims at characterizing a reachable set of a hybrid dynamical system with a lag constraint in the switch control. The setting does not consider any controllability assumptions and uses a level-set approach. The approach consists in the introduction of an adequate hybrid optimal control problem with lag constraints on the switch control whose value function allows a characterization of the reachable set. The value function is in turn characterized by a system of quasi-variational inequalities (SQVI). We prove a comparison principle for the SQVI which shows uniqueness of its solution. A class of numerical finite difference scheme for solving the system of inequalities is proposed and the convergence of the numerical solution towards the value function is studied using the comparison principle. Some numerical examples illustrating the method are presented. Our study is motivated by an industrial application. We are interested in the maximum range of hybrid vehicles.

Key words. Optimal control, Quasi-variational Hamilton-Jacobi equation, Hybrid systems, Reachability analysis

AMS subject classifications. 49LXX, 34K35, 34A38, 65M12

1. Introduction.

This paper deals with the characterization of a reachable set of a hybrid dynamical system with a lag constraint in the switch control. The approach consists in the introduction of an adequate hybrid optimal control problem with lag constraints on the switch control whose value function allows a characterization of the reachable set.

The term hybrid system refers to a general framework that can be used to model a large class of systems. Broadly speaking, they arise whenever a collection of continuous- and discrete-time dynamics are put together in a single model. In that sense, the discrete dynamics may dictate switching between the continuous dynamics, jumps in the system trajectory or both. Moreover, they can contain specificities, as for instance, autonomous jumps and/or switches, time delay between discrete decisions, switching/jumping costs. This work considers a particular class of hybrid systems where only switching between continuous dynamics are operated by the discrete logic, with no jumps in the trajectory, and with no switching costs. In addition, switch decisions are constrained to be separated in time by a non-zero interval, fact which is referred to as switching lag.

Before referring to the reachability problem in the hybrid setting, the main ideas are introduced in the non-hybrid framework. Given a time $t > 0$, a closed target set $\mathcal{C}$ and a closed admissible set K, consider a controlled dynamical system

$$\dot{y}(s) = f(s, y(s), u(s)), \quad \text{a.e. } s \in [0, t], \tag{1.1}$$

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where $f : \mathbb{R}^+ \times \mathbb{R}^d \times U \to \mathbb{R}^d$ and $u : \mathbb{R}^+ \to U$ is a measurable function. The reachable set $R_{\mathcal{C}}(t)$ is defined as the set of all positions x for which there exists a trajectory that starts in $\mathcal{C}$, stays inside K on $[0,t]$ and arrives in x at time t:

$$R_{\mathcal{C}}(t) := \{x \mid \exists (y,u) \text{ satisfies (1.1), with } y(0) \in \mathcal{C} \text{ and } y(s) \in K \text{ on } [0,t], \text{ and } y(t) = x\}.$$

It is a known fact that the reachable set can be characterized by the negative region of the value function of an optimal control problem. For this, following the idea introduced by Osher and Sethian [15], one can consider the control problem defined by:

$$v(x,t) := \inf\{v_0(y(0)) \mid (y,u) \text{ satisfies (1.1), with } y(t) = x \text{ and } y(s) \in K \text{ on } [0,t]\}, \tag{1.2}$$

where v_0 is a Lipschitz continuous function satisfying $v_0(x) \leq 0 \iff x \in \mathcal{C}$ (for instance, v_0 can be the signed distance $d_{\mathcal{C}}$ to $\mathcal{C}$). Under classical assumptions on the vector-field f, one can prove that the reachable set is given by

$$R_{\mathcal{C}}(t) = \{x \in K, v(x,t) \leq 0\}.$$

Moreover, when K is equal to $\mathbb{R}^d$, the value function has been shown to be the unique viscosity solution of a Hamilton-Jacobi-Bellman (HJB) equation [2] (for every $t > 0$):

$$\partial_t v + \sup_{u \in U} \{-f(s,x,u) \cdot \nabla v\} = 0 \quad \text{ on } \mathbb{R}^d \times (0,t],$$

with the initial condition $v(x,0) = v_0(x)$.

When the set K is a subset of $\mathbb{R}^d$ ($K \neq \mathbb{R}^d$), the characterization of v by means of a HJB equation becomes a more delicate matter. However, it was pointed out in [6] that in case of state-constraints, it is still possible to characterize the reachable set by an adequate control problem involving a supremum cost (see Section 2.3 for more details).

In this paper, we are interested in the extension of the reachability framework to some class of control problems of hybrid systems.

Let us recall that a hybrid dynamical system is a collection of controlled continuous-time processes selected through a high-level discrete control logic. A general framework for the (optimal) control hybrid dynamical systems was introduced in [7]. Several papers deal with the optimal control problem of hybrid systems, let us just mention here the papers [1,9,13,17] where the optimality conditions in the form of Pontryagin' principle are studied and [11,12,19] where the HJB approach is analyzed.

A feature of the hybrid system used in our work is a time lag between two consecutive switching decisions. From the mathematical viewpoint this removes the particularities linked to Zeno-like phenomena [18]. Indeed, the collection of state spaces is divided in subsets labeled in three categories according to whether they characterize discrete decisions as optional, required (autonomous) or forbidden. Landing conditions ensure that whatever the region of the state space the state vector "lands" after a switch no other switch is possible by requiring a positive distance (in the Hausdorff sense) between the landing sets and the optional/autonomous switch sets. In the other hand, when allowed to switch freely without any costs, when no time interval is imposed between discrete transitions, a controller with a possibly infinite number of instantaneous switches may become admissible. Switch costs can be introduced in order to rule out this kind of strategy by the controller as it becomes over-expensive

to switch to a particular mode using superfluous transitions. However, such costs do not make sense in the level-set approach used in this paper.

Diffusion processes with impulse controls including switch lags are studied in [8], where it is considered the idea of introducing a state variable to keep track of the time since the last discrete control decision. There, in addition, discrete decisions also suffer from a time delay before they can manifest in the continuous-time process. In that case, one has the possibility of scheduling discrete orders whenever the time for a decision to take place may be longer than that of deciding again. Then, the analysis also includes keeping record of the nature of this scheduled orders. This work inspired the idea of a state variable locking possible transitions used here.

To study the reachability sets for our system, we follow the level set approach and adapt the ideas developed in [6] to hybrid systems by proposing a suitable control problem which allows us to handle in a convenient way the state constraints and the decision lag. Here the main difficulty is to characterize the value function associated to the control problem. It turns out that this value function satisfies a quasi-variational HJB inequality system (in the viscosity sense). A comparison principle is derived for this system.

This paper is organized as follows: Section 2 states the associated hybrid optimal control problem and defines the reachable set and the value function. Section 3 presents the main results of the paper: it establishes a dynamic programming principle for the value function, shows that the value function is Lipschitz continuous and a solution of an system of quasi-variational inequalities (SQVI). It follows with the proof of a comparison principle for the SQVI that ensures uniqueness of its solution. Section 4 presents a numerical analysis of the SQVI. It shows the convergence of a class of numerical schemes for the computation of the value function and illustrates the convergence of a particular discretization scheme for a simple instance inspired by the vehicle applications.

2. Motivation and Problem Settings.

2.1. Hybrid Vehicles. A hybrid vehicle is a vehicle with two different sources of energy. The first one is a fuel tank and provides power through an internal combustion engine (ICE). The second one is a battery and provides power through an electric motor. Both the vehicle's energy sources are considered to have normalized energetic capacities – thus valued between 0 and 1.

The controls available to the controller are the ICE state – on or off – and the power produced in the ICE. The power produced in the ICE is a non-negative piecewise continuous function of time. The ICE's state is controlled by a discrete sequence of switching orders decided and executed at discrete times. The model assumes that the vehicle must stop whenever there is no charge left in the battery. Additionally, a feature of this model is a time interval $\delta > 0$ imposed between two consecutive decisions times. From the physical viewpoint, this assumption incorporates the fact that frequent switching of the ICE is undesirable in order to avoid mechanical wear off and acoustic nuisance for the driver. In this setting, the optimal controller seeks to best control the power of the ICE and its state to maximize the vehicle's autonomy, i.e. drive to the furthest point possible. The objective of finding the vehicle autonomy therefore translates in reaching the furthest point away from the vehicle geographic starting point where the battery is depleted for the first time.

2.2. Hybrid Dynamical System. Hybrid systems have some supervision logic that intervenes punctually between two or more continuous functions. The main

elements of the class of hybrid dynamical systems considered in this work are a discrete state q valued in a discrete set Q, and a family of continuous dynamics (vector fields) f and a continuous state space $X = \mathbb{R}^d$.

More precisely, the continuous state variable is denoted by y and it is valued in the state space X. The discrete variable is $q \in Q = \{0, 1, \cdots, d_q\}$, where d_q is the number of possible dynamics that can operate the system.

The continuous control is a measurable function $u : [0, \infty[\to \mathbb{R}^m$ valued in a set that depends on the discrete state $U(q)$. The discrete control is a sequence of switching decisions

$$w = \{(w_1, s_1), \cdots, (w_i, s_i), (w_{i+1}, s_{i+1}), \cdots\}, \tag{2.1}$$

where the s_i are *switching times* and the w_i are *switching decisions*. Switching times can happen at any time, therefore $s_i \in [0, \infty[$. Switching decisions are valued in a discrete set $W(q) \subset Q$ that depend on the discrete state, therefore $w_i \in W(q)$. Set $W(q)$ is the set of available orders when in discrete state q. The sequence of discrete switching decisions $\{w_i\}_{i>0}$ (designating the new mode of operation) is associated with a sequence of switching times $\{s_i\}_{i>0}$ where each decision w_i is exerted at time s_i. The case where no switch can occur corresponds to a single discrete input (w_1, s_1) where $s_1 = 0$ and $w_1 \in Q$ (this decision will have no effect on the controlled system as it is considered backward in time, see (2.5)). Denote by A the space of hybrid controls $a = (u, (w_i, s_i)_i)$.

The lag condition between switches is included as a control constraint that imposes two switch orders to be separated by a time interval of $\delta > 0$, i.e.,

$$s_{i+1} - s_i \geq \delta.$$

We precise the class of admissible controls $\mathcal{A} \subset A$ in the following definition:

DEFINITION 2.1. *For a fixed time horizon $T \geq 0$ a hybrid control $a = (u, (w_i, s_i)_i) \in \mathcal{A}$ is said to be admissible if the continuous control satisfies $u(s) \in U(q(s))$ and the discrete control sequence $w = \{w_i, s_i\}_{i\geq 1}$ has increasing decision times*

$$s_1 \leq s_2 \leq \cdots \leq s_i \leq s_{i+1} \leq \cdots \leq T, \tag{2.2a}$$

admissible decisions

$$\forall i > 0,\ w_i \in W(q(s_i)) \subset Q, \tag{2.2b}$$

and verifies a decision lag

$$s_{i+1} - s_i \geq \delta, \tag{2.2c}$$

where $\delta > 0$.

Let us point out that the above definition includes the case where no switch occurs during the interval time $(0, T]$, i.e. the sequence of switching times is reduced to one single element $s_1 = 0$.

An important consequence in the definition of admissible control is the finiteness of the number of switch orders:

PROPOSITION 2.2. *Fix $T \geq 0$. Let $a \in \mathcal{A}$ be an admissible hybrid control. Then, the discrete control sequence has at most $N = \lfloor T/\delta \rfloor$ switch decisions.*

Proof. The proof is trivial in a finite-time setting with imposed delay between switching orders. □

In this setting, given a discrete state $q(s) = q$, a *final* state x, a switching sequence (w_i, s_i) and a continuous control u. The system evolves according to

$$\dot{y}(s) = f(s, y(s), u(s), q(s)), \text{ for a.e. } s \in [0, t], \ y(t) = x. \tag{2.3}$$

When $q(s) = q_i$, the vector field corresponding to the discrete state q_i is active and dictates the evolution of the continuous state. At some isolated times $\{s_i\}_{i>0}$, given by the discrete switching control sequence, the *discrete dynamics* $g : Q^2 \to Q$ is activated (notice that the system is backward in time)

$$q_{i-1} = g(w_i, q_i) \tag{2.4}$$

and the continuous state follows another vector field $f(\cdot, \cdot, \cdot, q_{i-1})$. In the considered system, the discrete decisions only switch the continuous dynamics and introduce no discontinuities on the trajectory.

In more general frameworks, we can also include jumps in the continuous state vector that can be used to model an instant change in the value of the state following a discrete decisions [5, 7].

Assume controlled continuous dynamics f, the lag δ and the set of control values $U(q)$ satisfy the following assumptions:

(H0) Through all the paper the lag interval $\delta > 0$ is fixed. For every $q \in Q$, $U(q)$ is a compact set of $\mathbb{R}^m$ ($m \geq 1$).

(H1) There exists $L_f > 0$ such that, for all $s \geq 0$, $y, y' \in X$, $q \in Q$ and $u \in U(q)$,

$$\|f(s, y, u, q) - f(s, y', u, q)\| \leq L_f \|y - y'\|, \quad \|f(s, y, u, q)\| \leq L_f.$$

(H2) For all $q \in Q$, $f(\cdot, \cdot, \cdot, q) : [0, \infty[\times X \times \mathbb{R}^m \to X$ is continuous.

(H3) For all $s \geq 0$, $y \in X$ and $q \in Q$, $f(s, y, U(q), q)$ is a convex subset of X.

Assumption (H0),(H1) ensures that a trajectory exists and that it is unique. Assumption (H2) is used to prove the Lipschitz continuity of the value function and (H3) is needed in order to observe the compactness of the trajectory space.

Now we introduce the hybrid trajectory. Fix a time horizon $T > 0$ and let $0 < t < T$. Given a hybrid control $a = (u, (w_i, s_i)_i) \in A$ with *at most* N switch orders and given $x \in X$, $q \in Q$, the hybrid dynamical system is

$$\dot{y}(s) = f(s, y(s), u(s), q_i), \quad \text{for } s \in (s_{i-1}, s_i), \quad \text{and } y(t) = x, \tag{2.5a}$$

$$q(s) = q_i, \quad \text{for } s \in [s_{i-1}, s_i), \quad \text{and } q(t) = q, \tag{2.5b}$$

$$q_{i-1} = g(w_i, q_i), \quad i = 1, \cdots, N. \tag{2.5c}$$

Denote the solutions of (2.5a)-(2.5c) with final conditions x, q by $y_{x,q;t}$ and $q_{x,q;t}$. As pointed out, not all discrete control sequences are admissible. Only admissible control sequences engender admissible trajectories. Thus, given $t > 0$, $x \in X$ and $q \in Q$, the admissible trajectory set $Y^{x,q}_{[0,t]}$ is defined as

$$Y^{x,q}_{[0,t]} = \{y(\cdot) \mid a \in \mathcal{A} \text{ and } y_{x,q;t} \text{ solution of (2.5) } \} \tag{2.6}$$

A consequence of proposition 2.2 and the above definition is the finiteness of the number of discrete decisions in any admissible trajectory.

The hybrid control admissibility condition formulated as in conditions (2.2) is not well adapted to a dynamic programming principle formulation, needed later on. In order to include the admissibility condition in the optimal control problem in a more suitable form, we associate to each discrete control $(w_i, s_i)_i$ a new state variable π. Recall that the decision lag conditions implies that new switch orders are not available up to a time δ since the last switch. The new variable is constructed such that at a given time $s \in [0,t]$, the value of $\pi(s)$ measures the time since the last switch. The idea is to impose constraints on this new state variable and treat them more easily in the dynamic programming principle. Thus, if $\pi(s) < \delta$ all switch decisions are blocked and if, conversely, $\pi(s) \geq \delta$ the system is free to switch. For that reason, this variable can be seen as a switch lock.

More precisely, given $t > 0$, $s \in [0,t]$ and a discrete control $\{w_i, s_i\}_{i>0}$, the switch lock dynamics is defined by

$$\pi(s) = \begin{cases} \delta + s & \text{if} \quad s < s_1 \\ \inf_{s_i \leq s} s - s_i & \text{if} \quad s \geq s_1 \end{cases} \tag{2.7}$$

Indeed, once the discrete control is given, the trajectory $\pi(\cdot)$ can be determined. Proceeding with the idea of adapting the admissibility condition in order to manipulate it in a dynamic programming principle, we consider $\pi(t) = p$, with $p \in [0,T]$ *the final value* of the switch lock variable trajectory and impose the lag condition under the form $\pi(s_i^-) \geq \delta$ for all s_i, where s_i^- denotes the limit to the left at the switching times s_i (notice that $\pi(s_i^+) = 0$ by construction). Note that when a switch can occur, the final value of the switch lock p should be in $[0,t)$, while when no switch can occur, p can be considered as any value in $[t,T)$.

Then, since these conditions suffice to define an admissible discrete control set, while optimizing with respect to admissible functions, one needs only look within the set of hybrid controls that engenders a trajectory $\pi(\cdot)$ with the appropriate structure. In other words, given $t > 0$, $x \in X$, $q \in Q$ and $p \in [0,T)$, define a admissible trajectory set $S^{x,q,p}_{[0,t]}$ as

$$\begin{aligned} S^{x,q,p}_{[0,t]} = \{y(\cdot) \mid a = (u, \{w_i, s_i\}_{i=1}^N) \in A,\ y_{x,q,p;t} \text{ solution of (2.5a)-(2.5c)}, \\ \pi(\cdot) \text{ solution of (2.7)},\ \pi(t) = p,\ \pi(s_i^-) \geq \delta,\ i = 1, \cdots, N\}. \end{aligned} \tag{2.8}$$

The next lemma states a relation between sets Y and S:

LEMMA 2.3. *Set $T > 0$ is a final horizon. Following the above definitions, for every $t \in [0,T]$, sets* (2.6) *and* (2.8) *satisfy*

$$Y^{x,q}_{[0,t]} = \bigcup_{p \in [0,T)} S^{x,q,p}_{[0,t]}.$$

Proof. The equivalence between $Y^{x,q}_{[0,t]}$ and $\bigcup_{p\in[0,T)} S^{x,q,p}_{[0,t]}$ is obtained by construction. □

In the following of the paper, whenever we wish to call attention to the fact that the final conditions of (2.5a), (2.5c) and (2.7) are fixed, we denote their solutions respectively by $y_{x,q,p;t}, q_{x,q,p;t}, \pi_{x,q,p;t}$.

2.3. Reachability of Hybrid Dynamical Systems and Optimal Control Problem. Let $\mathcal{C} \subset X$ be the set of allowed initial states, i.e. the set of states from which the system (2.5a)-(2.5c) is allowed to start. Moreover, define a compact set

$K \subset X$ as the hybrid system admissible set, i.e., a set inside which the state must remain.

In the sequel $\mathcal{C}$ and K are assumed to be closed nonempty sets. Consider a finite horizon $T > 0$ and let $0 < t < T$. Define the reachable set as:

$$\begin{aligned} R_{\mathcal{C}}(t) &= \{x \mid \exists q \in Q,\ y \in Y^{x,q}_{[0,t]},\ y(0) \in \mathcal{C},\ \text{and } y(\theta) \in K\ \forall \theta \in [0,t]\} \\ &= \{x \mid \exists (q,p) \in Q{\times}[0,T),\ y \in S^{x,q,p}_{[0,t]},\ y(0) \in \mathcal{C},\ \text{and } y(\theta) \in K\ \forall \theta \in [0,t]\}. \end{aligned}$$

The reachable set $R_{\mathcal{C}}(t)$ contains the values of $y_{x,q;t}(t) = x$, regardless of the final discrete state, for all admissible trajectories – i.e., trajectories obtained through an admissible hybrid control – starting within the set of possible initial states $\mathcal{C}$, that never leave set K.

The following proposition ensures that the space of admissible trajectories is a compact set.

PROPOSITION 2.4. *Given $T > 0$ and assume (H1),(H2),(H3). Then, the admissible trajectory set $Y^{x,q}_{[0,T]}$ is a compact set in $C([0,T])$ endowed with the $W^{1,1}$-topology.*

Proof. Fix $q \in Q$ and $0 \le s < t \le T$. Consider a bounded admissible continuous control sequence $u_n \in L^1([s,t])$. Since u_n is bounded, there exists a subsequence u_{n_j} such that $u_{n_j} \rightharpoonup u$ in $L^1([s,t])$. Invoking (H1),(H2), we have $y^{u_n} = y_n \rightharpoonup y$ in $W^{1,1}([s,t])$. Since $W^{1,1}([s,t])$ is compactly embedded in $C^0([s,t])$, we get the strong convergence of the solution $y_n \to y$ in $C^0([s,t])$. Hypothesis (H3) guarantees that the limit function y is a solution of (2.5a). Because all controls u_n and the limit control u are admissible, y is an admissible solution.

So far, the proof shows that the limit trajectory is admissible when q is hold constant.

The goal of the following argument is to extend the previous reasoning to pieces of any trajectory, since q will eventually be constant outside switching times, due the finiteness of switches.

Consider a sequence of admissible discrete control sequences $(w)_n$ where the number of switching orders, $0 \le k_n \le \lfloor T/\delta \rfloor$ may depend on n. Since each term of this sequence has a (first) discrete component and is bounded on the (second) continuous component, then, as $n \to \infty$ there exists a subsequence $(w)_{n_l}$ and $\lambda > 0$ such that $q_{n_l} = q$ for all $l > \lambda$. This implies $k_n \to k$. As the number of switches is constant from the λ^{th} term and s, t are arbitrary, one can obtain, using the limit discrete control sequence w, the time intervals $[s_{i-1}, s_i]$ over which q_i is constant and the argument in the first paragraph of the proof. Because the trajectory is continuous and admissible on all time intervals $[s_{i-1}, s_i], i = 1, \cdots, I$, it is admissible on $[0,T]$. This completes the proof. □

REMARK 2.1. *The arguments presented in the above proof can be slightly modified to show that the admissible trajectory set with fixed final p, $S^{x,q,p}_{[0,T]}$ is compact. Also in a similar way, the proof can be adapted to show that the reachable $R_{\mathcal{C}}(t)$ is closed for any $t \ge 0$. Indeed, by the closedness of set $\mathcal{C}$, a sequence of initial conditions $(y_0)_n \in \mathcal{C}$, associated with admissible trajectories $y_n \in Y^{x,q}_{[0,T]}$, converges to $y_0 \in \mathcal{C}$ which is also the initial condition for the limiting trajectory $y_n \to y$.*

In order to characterize the reachable sets $R_{\mathcal{C}}(\cdot)$ this paper follows the classic level-set approach [15]. The idea is to describe (2.9) as the negative region of a function v. It is well known that the function v can be defined as the value function of some optimal control problem. In the case of system (2.5a)-(2.5c), v happens to be the value function of a hybrid optimal control problem.

Consider a bounded Lipschitz continuous function $\phi : X \to \mathbb{R}$ such that

$$\phi(x) \leq 0 \Leftrightarrow x \in \mathcal{C}. \tag{2.9}$$

Such a function always exists – for instance, the function

$$\phi(x) := \max(\min(d_{\mathcal{C}}(x), L_K), -L_K),$$

where $d_{\mathcal{C}}$ is the signed distance function from the set $\mathcal{C}$, and $L_K > 0$ a positive constant.

Define a bounded Lipschitz continuous function $\gamma : X \to \mathbb{R}$ to be

$$\gamma(x) \leq 0 \Leftrightarrow x \in K. \tag{2.10}$$

Such a function always exist as far as K is a closed set. In the sequel, we consider $L_K > 0$ a positive constant such that:

$$|\gamma(x)| \leq L_K \quad \text{and } |\phi(x)| \leq L_K, \quad \forall x \in X.$$

Then, for a given $t \geq 0$, $(x, q, p) \in X \times Q \times [0, T)$ and $y \in S^{x,q,p}_{[0,t]}$, define a total cost function to be

$$J(x, q, p, t; y) = \left(\phi(y(0)) \bigvee \max_{\theta \in [0,t]} \gamma(y(\theta))\right)$$

and then, the optimal value :

$$v(x, q, p, t) = \inf_{y \in S^{x,q,p}_{[0,t]}} J(x, q, p, t; y). \tag{2.11}$$

Observe that (2.11) is bounded thanks to the constructions (2.9) and (2.10). The idea in place is that one needs only to look at the sign of v to obtain information about the reachable set. Therefore, the bound L_K removes the necessity of dealing with an unbounded value function besides providing a natural bound for numerical computations.

3. Main Results. The next proposition certifies that (2.9) is indeed a level-set of (2.11).

PROPOSITION 3.1. *Assume (H1)-(H3). Define Lipschitz continuous functions ϕ and γ by* (2.9) *and* (2.10) *respectively. Define the value function v by* (2.11)*. Then, for $t \geq 0$, the reachable set is given by*

$$R_{\mathcal{C}}(t) = \{x \mid \exists (q, p) \in Q \times [0, T),\ v(x, q, p, t) \leq 0\} \tag{3.1}$$

Proof. Assume $y_{x,q,p;t}(t) \in R_{\mathcal{C}}(t)$. Then, by definition, there exists $(q, p) \in Q \times [0, T)$ and an admissible trajectory such that $y_{x,q,p;t}(\theta) \in K$ for all time and $y_{x,q,p;t}(0) \in \mathcal{C}$. This implies that $\max_{\theta \in [0,t]}(\gamma(y_{x,q,p;t}(\theta))) \leq 0$ and $\phi(y_{x,q,p;t}(0)) \leq 0$. It follows that $v(x, q, p, t) \leq J(x, q, p, t; y) \leq 0$.

Conversely, assume $v(x, q, p, t) \leq 0$. For any optimal trajectory $\hat{y}$ (which is admissible thanks to Proposition 2.4) $v(x, q, p, t) = J(x, q, p, t; \hat{y}) \leq 0$. Since the maximum of the two quantities is non positive only if they are both non positive one can draw the desired conclusion. ∎

Proposition (3.1) states that one can draw information about $R_{\mathcal{C}}(\cdot)$ by computing v. In the sequel, it is shown that the value function defined in (2.11) is the unique (viscosity) solution of a quasi-variational inequalities' system. The first step is to state a dynamic programming principle for (2.11).

First, we present some preliminary notation. Given $T > 0$, set $\Omega = X \times Q \times (0,T) \times (0,T]$ and denote its closure by $\overline{\Omega}$. For a fixed $p_0 \in [0,T)$, define $\Omega|_{p_0} = X \times Q \times \{p_0\} \times (0,T]$ and denote $\overline{\Omega}|_{p_0}$ the closure of $\Omega|_{p_0}$. Define

$$\mathcal{V}(\overline{\Omega}) := \{v \mid v : \overline{\Omega} \to \mathbb{R},\ v \text{ bounded }\}, \tag{3.2}$$

$$\mathcal{V}(\overline{\Omega}|_{p_0}) := \{v \mid\ v : \overline{\Omega}|_{p_0} \to \mathbb{R},\ v \text{ bounded }\},\ \text{for } p_0 \in [0,T). \tag{3.3}$$

For $v \in \mathcal{V}(\overline{\Omega})$, denote its upper and lower envelope at point $(x,q,p,t) \in \overline{\Omega}$ respectively as v^* and v_*:

$$v^*(x,q,p,t) = \limsup_{\substack{x_n \to x \\ q_n \to q \\ p_n \to p \\ t_n \to t}} v(x_n,q_n,p_n,t_n) \tag{3.4}$$

$$v_*(x,q,p,t) = \liminf_{\substack{x_n \to x \\ q_n \to q \\ p_n \to p \\ t_n \to t}} v(x_n,q_n,p_n,t_n) \tag{3.5}$$

In the case where $p_0 \in [0,T)$ is fixed and $v \in \mathcal{V}(\overline{\Omega}|_{p_0})$, the upper and lower envelopes of v are also given by (3.4), (3.5) with $p_n = p_0$ for all n.

Now, fix $p = 0$ and define the non-local switch operators $M, M^+, M^- : \mathcal{V}(\overline{\Omega}|_0) \to \mathcal{V}(\overline{\Omega}|_0)$ to be

$$(Mv)(x,q,0,t) = \inf_{\substack{w \in W(q) \\ p' \geq \delta}} v(x, g(w,q), p', t)$$

$$(M^+v)(x,q,0,t) = \inf_{\substack{w \in W(q) \\ p' \geq \delta}} v^*(x, g(w,q), p', t)$$

$$(M^-v)(x,q,0,t) = \inf_{\substack{w \in W(q) \\ p' \geq \delta}} v_*(x, g(w,q), p', t)$$

The action of these operators on the value function represents a switch that respects the lag constraint. They operate whenever a switch is activated, which is equivalent to the condition $p = 0$. Therefore, they are defined only for a fixed $p = 0$. In the following, denote $BUSC(\overline{\Omega})$ and $BUSC(\overline{\Omega})$ the set of bounded upper semi-continuous and bounded lower semi-continuous functions valued in $\overline{\Omega}$, respectively. Let us recall here some classical properties of operators M, M^+ and M^- (adapted from [19]):

LEMMA 3.2. *Let $v \in \mathcal{V}(\overline{\Omega})$. Then $M^+v^* \in BUSC(\overline{\Omega})$ and $M^-v_* \in BLSC(\overline{\Omega})$. Moreover $(Mv)^* \leq M^+v^*$ and $(Mv)_* \geq M^-v_*$.*

Proof. Fix $q \in Q$, $p = 0$ and $\epsilon > 0$. Let $w^* \in W(x,q)$ and $p^* > 0$ be such that for all $x \in X$ and $t \in (0,T)$, $(M^+v^*)(x,q,p,t) \geq v^*(x, g(w^*,q), p^*, t) - \epsilon$. Consider

sequences $x_n \to x$ and $t_n \to t$. Then

$$\begin{aligned} M^+v^*(x,q,0,t) &\geq v^*(x,g(w^*,q),p^*,t) - \epsilon \\ &\geq \limsup_{\substack{x_n\to x\\ t_n\to t}} v^*(x_n,g(w^*,q),p^*,t_n) - \epsilon \\ &\geq \limsup_{\substack{x_n\to x\\ t_n\to t}} \inf_{w\in W(x_n,q),p'\geq\delta} v^*(x_n,g(w,q),p',t_n) - \epsilon \\ &= \limsup_{\substack{x_n\to x\\ t_n\to t}} (M^+v^*)(x_n,q,0,t_n) - \epsilon. \end{aligned}$$

Notice that q and p are held constant throughout the inequalities and thus, the limsup of the jump operator considering only sequences $x_n \to x$ and $t_n \to t$ corresponds to its envelope at the limit point. Then, by the arbitrariness of ϵ, this proves the upper semi-continuity of M^+v^*. The lower semi-continuity of M^-v_* can be obtained in a similar fashion.

Now, observe that $Mv \leq M^+v^*$. Taking the upper envelope of each side, one obtains:

$$(Mv)^* \leq (M^+v^*)^* = M^+v^*.$$

By the same kind of reasoning $Mv \geq M^-v_*$ and

$$(Mv)_* \geq (M^-v_*)_* = M^-v_*.$$

□

The next proposition is the dynamic programming principle verified by (2.11):

PROPOSITION 3.3. *The value function* (2.11) *satisfies the following dynamic programming principle:*

(i) For t=0,

$$v(x,q,p,0) = \max(\phi(x),\gamma(x)), \quad \forall (x,q,p) \in X\times Q\times[0,T), \tag{3.6}$$

(ii) For $p = 0$,

$$v(x,q,0,t) = (Mv)(x,q,0,t), \quad (x,q,t) \in X\times Q\times[0,T], \tag{3.7}$$

(iii) For $(x,q,p,t) \in \Omega$, *define the non-intervention zone as* $\Sigma = (0, p\wedge t)$. *Then, for* $h \in \Sigma$,

$$v(x,q,p,t) = \inf_{S^{x,q,p}_{[t-h,t]}} \left\{ v(y_{x,q,p;h}(t-h),q,p-h,t-h) \bigvee \max_{\theta\in[t-h,t]} \gamma(y_{x,q,p;h}(\theta)) \right\} \tag{3.8}$$

Proof. The dynamic programming principle is composed of three parts.

(i): Equality (3.6) is obtained directly by definition (2.11).

(ii): " $\leq$ ". Let $(x,q,t) \in X\times Q\times[0,T]$ and $p = 0$. Consider an admissible hybrid control $a = (u(\cdot), \{w_i, s_i\}_{i=1}^N)$ and construct a new control $\overline{a} = (\overline{u}(\cdot), \{\overline{w}_i, \overline{s}_i\}_{i=1}^{N-1})$ with associated trajectory $y^{\overline{a}}$, where $\overline{u} = u$, $\overline{w}_i = w_i$ and $\overline{s}_i = s_i$ for $i = 1,\cdots,N-1$ and $s_N = t$, $w_N = w'$. Then, one obtains:

$$v(x,q,0,t) \leq J(x,q,0,t;y^a) = J(x,g(w',q),p',t;y^{\overline{a}}),$$

where the controller must respect the condition $p' \geq \delta$ for it to be admissible. Since $\overline{a}$ is arbitrary, then we get:

$$v(x,q,0,t) \leq \inf_{y^{\overline{a}} \in S^{(x,g(w',q),p')}_{[0,t]}} J(x,g(w',q),p',t;y^{\overline{a}}) = v(x,g(w',q),p',t)$$

for every $p' \geq \delta$, $w' \in W(q)$. This yields to

$$\begin{aligned} v(x,q,0,t) &\leq \inf_{\substack{w' \in W(q) \\ p' \geq \delta}} v(x,g(w',q),p',t) \\ &= (Mv)(x,q,p,t). \end{aligned}$$

" $\geq$ ". For $(x,q) \in X \times Q$ and $p = 0$, for $\epsilon > 0$, there exists an admissible control a_ϵ, such that

$$v(x,q,0,t) + \epsilon \geq J(x,q,0,t;y^{a_\epsilon}).$$

Using the same hybrid control constructions as in the "$\leq$" case, one obtains

$$\begin{aligned} J(x,q,0,t;y^{a_\epsilon}) &= J(x,g(w',q),p',t;y^{\bar{a}_\epsilon}) \\ &\geq v(x,g(w',q),p',t) \\ &\geq \inf_{\substack{w' \in W(q) \\ p' \geq \delta}} v(x,g(w',q),p',t) = (Mv)(x,q,0,t). \end{aligned}$$

Relation (3.7) is obtained by the arbitrariness of ϵ.

(iii): " $\leq$ ". For $(x,q,p,t) \in \Omega$ and $0 < h \leq p \wedge s$, (2.11) yields

$$v(x,q,p,t) \leq \max\left(\left(\phi(y_{x,q,p;t}(0)) \bigvee \max_{\theta \in [0,t-h]} \gamma(y_{x,q,p;t}(\theta))\right), \max_{\theta \in [t-h,t]} \gamma(y_{x,q,p;t}(\theta))\right), \tag{3.9}$$

for any $y \in S^{x,q,p}_{[0,t]}$. By the choice of h, there is no switching between times $t-h$ and t. Write the admissible control $a = (u,w)$ as $a_0 = (u_0,w)$ and $a_1 = (u_1,w)$ with

$$\begin{aligned} u_0(s) &= u(s), \quad s \in [0,t-h], \\ u_1(s) &= u(s), \quad s \in (t-h,t]. \end{aligned}$$

Since a is admissible, both controls a_0, a_1 are also admissible. Denote the trajectory associated with controls a, a_0, a_1 respectively by y^a, y^0, y^1. Then, $y^a \in S^{x,q,p}_{[0,t]}$ and by continuity of the trajectory we achieve the following decomposition:

$$y^1 \in S^{x,q,p}_{[t-h,t]}, \qquad y^0 \in S^{y^1(t-h),q,p-h}_{[0,t-h]}.$$

The above decomposition together with inequality (3.9) yields

$$v(x,q,p,t) \leq \max\left(\left(\phi(y^0(0)) \bigvee \max_{\theta \in [0,s-h]} \gamma(y^0(\theta))\right), \max_{\theta \in [t-h,t]} \gamma(y^1(\theta))\right), \tag{3.10}$$

And one concludes after minimizing with respect to the trajectories associated with a_0 and a_1.

The " $\geq$ " part uses an ϵ-optimal controller and the same decomposition to obtain the inverse inequality of (3.10), allowing to conclude by the arbitrariness of ϵ. This is possible because there is no switching between $t-h$ and t. □

A direct consequence of proposition 3.3 is the Lipschitz continuity of the value function, stated in the next proposition:

PROPOSITION 3.4. *Assume (H1)-(H2). Define Lipschitz continuous functions ϕ and γ by* (2.9) *and* (2.10)*, with Lipschitz constants L_ϕ and L_γ respectively. Then, for any $T>0$, the value function defined in* (2.11) *is Lipschitz continuous on $X\times Q\times[0,T)\times[0,T]$.*

Proof. Fix $t\geq 0$, $x,x'\in X$, $q\in Q$, $p>0$. Then, using $\max(A,B)-\max(C,D)\leq \max(A-B,C-D)$, one obtains,

$$\begin{aligned}
|v(x,q,p,t)-v(x',q,p,t)| &\leq \max\Big(|\phi(y_{x,q,p;t}(0))-\phi(y_{x',q,p;t}(0))|, \\
&\qquad \max_{\theta\in[0,t]}|\gamma(y_{x,q,p;t}(\theta))-\gamma(y_{x',q,p;t}(\theta))|\Big) \\
&\leq \max\Big(L_\phi\,|y_{x,q,p;t}(0)-y_{x',q,p;t}(0)|, \\
&\qquad L_\gamma \max_{\theta\in[0,t]}(|y_{x,q,p;t}(\theta)-y_{x',q,p;t}(\theta)|)\Big) \\
&\leq L_v|x-x'|,
\end{aligned}$$

where $L_v=\max(L_\phi,L_\gamma)e^{L_f T}$.

Now, take $h>0$ and observe that $v(x,q,p,t)\geq\gamma(x)$. Then,

$$\begin{aligned}
|v(x,q,p+h,t+h)-v(x,q,p,t)| &\leq \max\Big(\Big|v(y_{x,q,p+h;t+h}(t),q,p,t)-v(x,q,p,t)\Big|, \\
&\qquad \Big|\max_{\theta\in[t,t+h]}\gamma(y_{x,q,p;t+h}(\theta))-\gamma(x)\Big|\Big) \\
&\leq \max\Big(L_v\Big|y_{x,q,p+h;t+h}(t)-x\Big|, \\
&\qquad L_\gamma \max_{\theta\in[t,t+h]}|y_{x,q,p+h;t+h}(\theta)-x|\Big) \\
&\leq L_f L_v h.
\end{aligned}$$

□

In order to proceed to the HJB equations, define the Hamiltonian to be

$$H(t,x,q,z)=\sup_{u\in U(q)} f(t,x,u,q)\cdot z. \tag{3.11}$$

Before stating the next result, we recall the notion of viscosity solution [10] used throughout this paper for the equation:

$$\partial_t v+\partial_p v+H(t,x,q,\nabla_x v)\bigwedge v-\gamma(x)=0 \quad \text{if } (x,q,p,t)\in\Omega \tag{3.12a}$$

$$v(x,q,p,t)=(M^+v)(x,q,p,t) \quad \text{if } p=0, \tag{3.12b}$$

$$v(x,q,p,t)=\max(\phi(x),\gamma(x)) \quad \text{if } t=0, \tag{3.12c}$$

DEFINITION 3.5. *A function u_1 (resp. u_2) upper semi-continuous (u.s.c.) (resp. lower semi-continuous (l.s.c)) is a viscosity subsolution (resp. supersolution) of* (3.12) *if for every continuously differentiable function ψ such that $u_1-\psi$ has a local maximum (resp. $u_2-\psi$ has a local minimum) at $(x,q,p,t)\in\overline{\Omega}$ we have:*

$$\begin{aligned}\partial_t\psi+\partial_p\psi+H(t,x,q,\nabla_x\psi)\bigwedge u_1-\gamma(x)\le 0 \quad & \text{if } (x,q,p,t)\in\Omega\\ u_1(x,q,p,t)\le (M^+u_1)(x,q,p,t) \quad & \text{if } p=0,\\ u_1(x,q,p,t)\le \max(\phi(x),\gamma(x)) \quad & \text{if } t=0,\end{aligned}$$

(with the inequalities signs inversed and M^- instead of M^+ for u_2). A bounded function u is a (viscosity) solution of (3.12) *if u^* is a subsolution and u_* is a supersolution (where, we recall, u^* is the upper semi-continuous envelope and u_* the lower semi-continuous envelope of u).*

The next statement shows that the value function defined in (2.11) is a solution of the quasi-variational system (3.12).

THEOREM 3.6. *Assume (H1)-(H2). Let T be a given finite horizon. Let the Lipschitz functions ϕ and γ be defined by* (2.9) *and* (2.10) *respectively. Then, the Lipschitz, bounded value function v defined in* (2.11) *is a viscosity solution of the quasi-variational inequality* (3.12) *on $\Omega=X\times Q\times(0,T)\times(0,T]$.*

Proof. By definition, v satisfies the initial condition (3.12c). The boundary condition (3.12b) is deducted from proposition 3.3. Now, we proceed to show that (i) v is a supersolution and (ii) a subsolution of (3.12a):

First, let us prove the supersolution property (i). To satisfy $\min(A,B)\ge 0$ one needs to show $A\ge 0$ and $B\ge 0$. Since $v-\gamma(x)\ge 0$, it is immediate that $B\ge 0$. Now, consider $0<h\le p\wedge t$. Let ψ be a continuously differentiable function such that $v-\psi$ attains a minimum at (x,q,p,t). Then, using proposition 3.3-(iii) and selecting an ϵ-optimal controller, dependent on h, with associated trajectory $y^{\epsilon}_{x,q,p;t}$, it follows that

$$\begin{aligned}\psi(x,q,p,t)=v(x,q,p,t)&\ge \inf_{S^{x,q,p}_{[0,t]}} v(y_{x,q,p;t}(t-h),q,p-h,t-h)\\ &\ge v(y^{\epsilon}_{x,q,p;t}(t-h),q,p-h,t-h)-h\epsilon\\ &= \psi(y^{\epsilon}_{x,q,p;t}(t-h),q,p-h,t-h)-h\epsilon\end{aligned}$$

and then,

$$\psi(x,q,p,t)-\psi(y^{\epsilon}_{x,q,p;t}(t-h),q,p-h,t-h)\ge -h\epsilon.$$

Since the control domain is bounded and using the continuity of f and ψ we divide by h and take the limit $h\to 0$ to obtain

$$\partial_t\psi+\partial_p\psi+H(t,x,q,\nabla_x\psi)\ge-\epsilon$$

and conclude that $A\ge 0$ by the arbitrariness of ϵ.

For (ii), observe that for $\min(A,B)\le 0$ it suffices to show that $A\le 0$ or $B\le 0$. If $v(x,q,p,t)=\gamma(x)$, it implies $B\le 0$. On the contrary, if $v(x,q,p,t)>\gamma(x)$, then there exists a $\Sigma\ni h\ge 0$ small enough so that

$$v(y^{u}_{x,q,p;t}(t-h),q,p-h,t-h)>\max_{\theta\in[t-h,h]}\gamma(y^{u}_{x,q,p;t}(\theta))$$

strictly, using the Lipschitz continuity of f, p and the compactness of U (which ensures the trajectories will remain near each other). Thus, proposition 3.3(iii) yields

$$v(x,q,p,t) = \inf_{y^u \in S^{x,q,p}_{[t-h,t]}} v(y^u_{x,q,p;t}(t-h), q, p-h, t-h).$$

Let ψ be a continuously differentiable function such that $v - \psi$ attains a maximum at (x,q,p,t). Fix an arbitrary $u \in U$ and consider a constant control $u(s) = u$ for $t-h < s < t$. Also, without loss of generality, assume that $v(x,q,p,t) = \psi(x,q,p,t)$. Hence,

$$\begin{aligned} v(x,q,p,t) &\leq v(y^u_{x,q,p;t}(t-h), q, p-h, t-h) \\ &\leq \psi(y^u_{x,q,p;t}(t-h), q, p-h, t-h) \end{aligned}$$

and by dividing by h and taking $h \to 0$ one obtains

$$\partial_t \psi + \partial_p \psi + f(t,x,u) \cdot \nabla_x \psi \leq 0.$$

Since u is arbitrary and admissible, we conclude that $A \leq 0$, which completes the proof. □

Theorem 3.6 provides a convenient way to characterize the value function whose level-set is the reachable set defined in (2.9). However, in order to be sure that the solution that stems from (3.12) corresponds to (2.11), a uniqueness result is necessary. This is achieved by a comparison principle which is stated in the next theorem. We recall that $BUSC(\Omega)$ and $BLSC(\Omega)$ respectively are the space of bounded u.s.c. and bounded l.s.c. functions defined over the set Ω.

THEOREM 3.7. *Assume (H1)-H2). Let T be a given finite horizon and denote $\Omega = X \times Q \times (0,T) \times (0,T]$. Let $u_1 \in BUSC(\Omega)$ and $u_2 \in BLSC(\Omega)$ be, respectively, sub- and supersolution of* (3.12). *Then, $u_1 \leq u_2$ in $\overline{\Omega}$.*

The proof is inspired by earlier work on uniqueness results for hybrid control problems. The idea is to show that $u_1 \leq u_2$ in all domain Ω and then on the boundary $p = 0$. The main difficulty arises when dealing with points in the boundary $p = 0$ where the system has a switching condition given by a non-local switch operator. This is tackled by the utilization of "friendly giant"-like test functions [3], [14]. Classically, these functions are used to prove uniqueness for elliptic problems with unbounded value functions where they serve to localize some arguments regardless of the function's possible growth at infinity. This feature proves itself very useful in our case because one can properly split the domain in no-switching and switching regions. In this work, the lag condition for the switch serves as an equivalent to the "landing condition"– which states that after an autonomous switch the system must land at some positive distance away from the autonomous switch set [11], [19].

Proof. Let Ω be defined as above, $\partial\Omega|_T = X \times Q \times (0,T) \times \{0\}$ and $\partial\Omega|_P = X \times Q \times \{0\} \times (0,T]$.

First, the comparison principle is proved for $\partial\Omega|_T$ (case 1), followed by Ω (case 2) and finally for $\partial\Omega|_P$ (case 3), which concludes the proof for $\overline{\Omega}$.

Case 1: At a point $(x,q,p,t) \in \partial\Omega|_T$, from the sub- and supersolution properties,

$$\begin{aligned} u_1(x,q,p,0) - \max(\phi(x), \gamma(x)) &\leq 0, \\ -u_2(x,q,p,0) + \max(\phi(x), \gamma(x)) &\leq 0, \end{aligned}$$

which readily yields $u_1 \leq u_2$ in $\partial\Omega|_T$.

Case 2: Start by using to sub- and supersolution properties of u_1, u_2 to obtain, in Ω,

$$\min(\partial_t u_1 + \partial_p u_1 + H(t,x,q,\nabla_x u_1), u_1 - \gamma(x)) \leq 0, \tag{3.13}$$
$$\min(\partial_t u_2 + \partial_p u_2 + H(t,x,q,\nabla_x u_2), u_2 - \gamma(x)) \geq 0. \tag{3.14}$$

Expression (3.14) implies that both

$$u_2 \geq \gamma(x) \tag{3.15}$$

and

$$\partial_t u_2 + \partial_p u_2 + H(t,x,q,\nabla_x u_2) \geq 0. \tag{3.16}$$

From (3.13), one has to consider two possibilities. The first one is when $u_1 \leq \gamma(x)$. If so, together with (3.15), one has immediately $u_1 \leq u_2$. Now, if $\partial_t u_1 + \partial_p u_1 + H(t,x,q,\nabla_x u_1) \leq 0$, one turns to (3.16).

Define $v = u_1 - u_2$. Notice that $v \in BUSC(\Omega)$. The next step is to show that v is a subsolution of

$$\partial_t v + \partial_p v + H(t,x,q,\nabla_x v) = 0 \tag{3.17}$$

at $(\bar{x}, \bar{q}, \bar{p}, \bar{t})$.

Let $\psi \in C^2(\Omega)$, bounded, be such that $v - \psi$ has a strict local maximum at $(\bar{x}, \bar{q}, \bar{p}, \bar{t}) \in \Omega$. Define auxiliary functions over $\Omega^i \times \Omega^i$, $i = 0, 1$ as

$$\Phi^i_\epsilon(x,p,t,\xi,\pi,\varsigma) = u_1(x,i,p,t) - u_2(\xi,i,\pi,\varsigma) - \psi(x,p,t) - \frac{|x-\xi|^2}{2\epsilon} - \frac{|p-\pi|^2}{2\epsilon} - \frac{|t-\varsigma|^2}{2\epsilon}. \tag{3.18}$$

Because the boundedness of ψ, u_1 and u_2 the suprema points are finite, for each $i = 0, 1$. Denote $(x_\epsilon, p_\epsilon, t_\epsilon, \xi_\epsilon, \pi_\epsilon, \varsigma_\epsilon) \in \Omega^{\bar{q}} \times \Omega^{\bar{q}}$ a point such that

$$\Phi^{\bar{q}}_\epsilon(x_\epsilon, p_\epsilon, t_\epsilon, \xi_\epsilon, \pi_\epsilon, \varsigma_\epsilon) = \sup_{\Omega^{\bar{q}} \times \Omega^{\bar{q}}} \Phi^{\bar{q}}_\epsilon(x,p,t,\xi,\pi,\varsigma).$$

The following lemma (proved further below for readability) establishes some estimations needed further in the proof:

LEMMA 3.8. *Define Φ^i_ϵ and $(x_\epsilon, p_\epsilon, t_\epsilon, \xi_\epsilon, \pi_\epsilon, \varsigma_\epsilon)$ as above. Then, as $\epsilon \to 0$,*

$$\frac{|x_\epsilon - \xi_\epsilon|^2}{\epsilon} \to 0, \quad \frac{|p_\epsilon - \pi_\epsilon|^2}{\epsilon} \to 0, \quad \frac{|t_\epsilon - \varsigma_\epsilon|^2}{\epsilon} \to 0,$$
$$|x_\epsilon - \xi_\epsilon| \to 0, \quad |p_\epsilon - \pi_\epsilon| \to 0, \quad |t_\epsilon - \varsigma_\epsilon| \to 0,$$

and $(x_\epsilon, p_\epsilon, t_\epsilon, \xi_\epsilon, \pi_\epsilon, \varsigma_\epsilon) \to (\bar{x}, \bar{p}, \bar{t}, \bar{x}, \bar{p}, \bar{t})$

A straightforward calculation allows to show that there exists $a, b \in \mathbb{R}$ such that

$$(a, b, D_\epsilon) \in D^- u_2(\xi_\epsilon, \bar{q}, \pi_\epsilon, \varsigma_\epsilon)$$
$$(a + \partial_t \psi, b + \partial_p \psi, D_\epsilon + \nabla_x \psi) \in D^+ u_1(x_\epsilon, \bar{q}, p_\epsilon, t_\epsilon),$$

where D^-, D^+ respectively denote the sub- and super differential [10] and $D_\epsilon = 2|x_\epsilon - \xi_\epsilon|/\epsilon$, which implies

$$a + b + H(\varsigma_\epsilon, \xi_\epsilon, \bar{q}, D_\epsilon) \geq 0$$
$$a + \partial_t \psi + b + \partial_p \psi + H(t_\epsilon, x_\epsilon, \bar{q}, D_\epsilon + \nabla_x \psi) \leq 0,$$

which in turn yields, as $\epsilon \to 0$,

$$\partial_t \psi + \partial_p \psi - L_f |\nabla_x \psi| \leq 0$$

at $(\bar{x}, \bar{q}, \bar{p}, \bar{t}) \in \Omega$. By adequately choosing the test functions ψ, one can repeat the arguments to show that this assertion holds for any point in Ω. Thus, this establishes that v is a subsolution of (3.17) in Ω.

Now, take $\kappa > 0$ and define a non-decreasing differentiable function $\chi_\kappa : (-\infty, 0) \to \mathbb{R}^+$ such that

$$\chi_\kappa(x) = 0,\ x \leq -\kappa\ ;\ \chi_\kappa(x) \to \infty,\ x \to 0.$$

Take $\eta > 0$, and define a test function

$$\nu(x, p, t) = \eta t^2 + \chi_\kappa(-p).$$

Observe that $v - \nu$ achieves a maximum at a finite point $(x_0, \bar{q}, p_0, t_0) \in \Omega$. Since κ can be made arbitrarily small one can consider $p_0 > \kappa$ without loss of generality. Therefore, using the subsolution property of v, by a straightforward calculation one has

$$2\eta t_0 \leq 0,$$

since $\chi'_\eta(-p_0) = 0$. The above inequality implies that $t_0 = 0$. Noticing that $\nu(x_0, p_0, t_0) = v(x_0, \bar{q}, p_0, t_0) = 0$, it follows

$$v(x, \bar{q}, p, t) \leq \eta t^2 + \chi_\kappa(-p)$$

for all $t \in (0, T]$, $x \in X$ and $p > \kappa$. Letting $\eta \to 0$, $\kappa \to 0$ and from the arbitrariness of $\bar{q}$, we conclude that $v \leq 0$ in Ω.

Case 3: In this case the switch lock variable arrives at the boundary of the domain, incurring thus a switch, as all others variables remain inside the domain. For all $(x_0, q_0, p_0, t_0) \in \partial\Omega|_P$, for any $p \geq \delta$ one has (using case 2 and noticing that $M^+ u_1 = M u_1$ and $M^- u_2 = M u_2$)

$$(M^+ u_1)(x_0, q_0, p_0, t_0) \leq u_1(x_0, q_0, p, t_0) \leq u_2(x_0, q_0, p, t_0).$$

Taking the infimum with respect to p, the above expression yields $M^+ u_1 \leq M^- u_2$ in $\partial\Omega|_P$. This suffices to conclude, since that by the sub- and supersolution properties

$$v = u_1 - u_2 \leq M^+ u_1 - M^- u_2.$$

□

Now we present the proof of Lemma 3.8:

Proof. Writing

$$2\Phi^i_\epsilon(x_\epsilon, p_\epsilon, t_\epsilon, \xi_\epsilon, \pi_\epsilon, \varsigma_\epsilon) \geq \Phi^i_\epsilon(x_\epsilon, p_\epsilon, t_\epsilon, x_\epsilon, p_\epsilon, t_\epsilon) + \Phi^i_\epsilon(\xi_\epsilon, \pi_\epsilon, \varsigma_\epsilon, \xi_\epsilon, \pi_\epsilon, \varsigma_\epsilon),$$

for $i = 0, 1$, one obtains,

$$\frac{|x_\epsilon - \xi_\epsilon|^2}{\epsilon} + \frac{|p_\epsilon - \pi_\epsilon|^2}{\epsilon} + \frac{|t_\epsilon - \varsigma_\epsilon|^2}{\epsilon} \leq (u_1 + u_2)(x_\epsilon, i, p_\epsilon, t_\epsilon) - (u_1 + u_2)(\xi_\epsilon, i, \pi_\epsilon, \varsigma_\epsilon) + \psi(x_\epsilon, p_\epsilon, t_\epsilon) - \psi(\xi_\epsilon, \pi_\epsilon, \varsigma_\epsilon),$$

which means that, since ψ, u_1 and u_2 are bounded that

$$\frac{|x_\epsilon - \xi_\epsilon|^2}{\epsilon} \leq C^*, \quad \frac{|p_\epsilon - \pi_\epsilon|^2}{\epsilon} \leq C^*, \quad \frac{|t_\epsilon - \varsigma_\epsilon|^2}{\epsilon} \leq C^*, \tag{3.19}$$

where C^* depends on the $\sup|u_1|$, $\sup|u_2|$, $\sup|\psi|$ and is independent of ϵ. Expression (3.19) yields

$$|x_\epsilon - \xi_\epsilon| \leq \sqrt{\epsilon C^*}, \quad |p_\epsilon - \pi_\epsilon| \leq \sqrt{\epsilon C^*}, \quad |t_\epsilon - \varsigma_\epsilon| \leq \sqrt{\epsilon C^*}.$$

which implies that the doubled terms tend to zero.

Since $(\bar{x}, \bar{q}, \bar{p}, \bar{t})$ is a strict maximum of $v - \psi$, one gets $(x_\epsilon, p_\epsilon, t_\epsilon, \xi_\epsilon, \pi_\epsilon, \varsigma_\epsilon) \to (\bar{x}, \bar{p}, \bar{t}, \bar{x}, \bar{p}, \bar{t})$. Remark that, since $\bar{p} > 0$, one can always choose a suitable subsequence $\epsilon_n \to 0$ such that all $p_{\epsilon_n} > 0$, avoiding thus touching the switching boundary. □

4. Numerical Analysis.

4.1. Numerical Scheme and Convergence. Equations (3.12) can be solved using a finite difference scheme on a domain $\Omega = X \times Q \times (0,T) \times (0,T]$, for $T > 0$. This section proposes a class of discretization schemes and shows its convergence using the Barles-Souganidis [4] framework.

Set mesh sizes $\Delta x > 0$, $\Delta p > 0$, $\Delta t > 0$ and denote the discrete grid point by (x_I, p_k, t_n), where $x_I = I\Delta x$ with $I \in \mathbb{Z}^d$, $p_k = k\Delta p$ with $k = 0, \cdots, n_p$, and $t_n = n\Delta t$ with $n = 0, \cdots, n_t$ where $n_p = T/\Delta p$ and $n_t = T/\Delta t$. The approximation of the value function is denoted by

$$v(x_I, q, p_k, t_n) = v^n_{Ik}(q)$$

and the penalization functions are denoted by $\phi(x_I) = \phi_I$, $\gamma(x_I) = \gamma_I$. Define the following grids:

$$G^\# = \Delta x\mathbb{Z}^d \times Q \times \Delta p\{0, 1, \cdots, n_p\} \times \Delta t\{0, 1, \cdots, n_t\},$$
$$G^\#_p = \Delta t\{0, 1, \cdots, n_t\} \times \Delta x\mathbb{Z}^d \times Q,$$

and the discrete space gradient at point x_I for any general function μ:

$$D^\pm\mu(X_I) = D^\pm\mu_I = \left(D^\pm_{x_1}\mu_I, \cdots, D^\pm_{x_d}\mu_I\right),$$

where

$$D^\pm_{x_j}\mu_I = \pm\frac{\mu_{I^{j,\pm}} - \mu_I}{\Delta x},$$

with

$$I^{j,\pm} = (i_1, \cdots, i_{j-1}, i_j \pm 1, i_{j+1}, \cdots, i_d).$$

Define a numerical Hamiltonian $\mathcal{H} : G^\#_p \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ destined to be an approximation of H. We assume that $\mathcal{H}$ verifies the following hypothesis:

(H4) There exists $L_{H_1}, L_{H_2} > 0$ such that, for all $t, x, q \in G^\#_p$ and $A^+, A^-, B^+, B^- \in \mathbb{R}^d$,

$$|\mathcal{H}(t,x,q,A^+,A^-) - \mathcal{H}(t,x,q,B^+,B^-)| \leq L_{H_1}(||A^+ - B^+|| + ||A^- - B^-||)$$
$$||\mathcal{H}(t,x,q,A^+,A^-)|| \leq L_{H_2}(||A^+ + A^-||).$$

(H5) The Hamiltonian satisfies the monotonicity condition for all $t, x, q \in G_p^{\#}$ and almost every $A^+, A^- \in \mathbb{R}^d$:

$$\partial_{A_i^+}\mathcal{H}(t,x,q,A^+,A^-) \leq 0, \text{ and } \partial_{A_i^-}\mathcal{H}(t,x,q,A^+,A^-) \geq 0.$$

(H6) There exists $L_{H_3} > 0$ such that for all $t, x, q \in G_p^{\#}$, $t', x', q' \in [0, \Delta t n_t] \times X \times Q$ and $A \in \mathbb{R}^d$,

$$|\mathcal{H}(t,x,q,A,A) - H(t',x',q',A)| \leq L_{H_3}(|t-t'| + ||x-x'|| + |q-q'|).$$

Let $\Phi : \Omega \to \mathbb{R}$, $h = (\Delta x, \Delta p, \Delta t)$ and set

$$S_h^{\Omega}(x,q,p,t,\lambda;\Phi) = \min\Big(\lambda - \gamma(x), \mathcal{H}(t,x,q,D^+\Phi(x,q,p,t-\Delta t), D^-\Phi(x,q,p,t-\Delta t)) + \frac{\lambda - \Phi(x,q,p,t-\Delta t)}{\Delta t} + \frac{\lambda - \Phi(x,q,p-\Delta p,t-\Delta t)}{\Delta p}\Big).$$

Now, consider the following operators:

$$S_h(x,q,p,t,\lambda;\Phi) := \begin{cases} S_h^{\Omega}(x,q,p,t,\lambda;\Phi) & \text{if} \quad (x,q,p,t) \in \Omega \\ \lambda - \min_{w \in W(x,q), p' \geq \delta} \Phi(x, g(w,q), p', t) & \text{if} \quad p = 0, \end{cases} \tag{4.1}$$

and

$$\mathcal{F}(x,q,p,t,u,\nabla u) := \begin{cases} u - \gamma(x) \bigwedge \partial_t u + \partial_p u + H(t,x,q,\nabla_x u) & \text{if} \quad (x,q,p,t) \in \Omega \\ u(x,q,p,t) - (Mu)(x,q,p,t) & \text{if} \quad p = 0. \end{cases} \tag{4.2}$$

With these notations, the equation (3.12) is equivalent to

$$\mathcal{F}(x,q,p,t,v,\nabla v) = 0 \qquad \text{in } \Omega.$$

We define an approximation scheme by:

$$S_h(x,q,p,t,u_h(x,q,p,t),u_h) = 0 \quad \text{ in } G^{\#}. \tag{4.3}$$

PROPOSITION 4.1. *Let $\Phi \in C_b^{\infty}(\Omega)$. Under hypothesis (H1)-(H2), (H4)-(H6) and the CFL condition*

$$\Delta t \left(\frac{1}{\Delta p} + \frac{1}{\Delta x} \sum_{i=1}^{d} \partial_{A_i^+}\mathcal{H} + \partial_{A_i^-}\mathcal{H} \right) \leq 1 \tag{4.4}$$

the discretization scheme (4.3) *is stable, monotone and consistent.*

Moreover, u_h converges towards the function v, as $h \to 0$.

Proof. The proof follows the lines used in the framework of Barles-Souganidis [4] inside Ω. We then complete the proof with the case $p = 0$. The goal is to show that the numerical scheme solutions' envelopes

$$\underline{u}(x',q',p',t') = \liminf_{\substack{(x,q,p,t) \to (x',q',p',t') \\ h \to 0}} u_h(x,q,p,t)$$
$$\overline{u}(x',q',p',t') = \limsup_{\substack{(x,q,p,t) \to (x',q',p',t') \\ h \to 0}} u_h(x,q,p,t),$$

are respectively supersolution and subsolution of (4.2). Then, using the comparison principle in theorem 3.7, one obtains $\overline{u} \leq \underline{u}$. However, since the inverse inequality is immediate (using the definition of limsup and liminf, one gets $u \equiv \underline{u} = \overline{u}$ achieving thus the convergence.

Only the subsolution property of $\overline{u}$ is presented next, the proof of the supersolution property of $\underline{u}$ being very alike.

Inside Ω it is sufficient to show that S_h^Ω is stable, monotone and consistent. Observe that S_h is proportional to $-\Phi$ in the terms outside the Hamiltonian. Since fluxes $\mathcal{H}$ are monotone by hypothesis $(H7)$ (see [16] for details in monotone Hamiltonian fluxes) whenever the CFL condition (4.4) is satisfied, the monotonicity of S_h follows. The stability is ensured by the boundedness of Φ and hypothesis $(H6)$. Finally, hypothesis $(H8)$ and lemma 3.2 are used in a straightforward fashion to obtain the consistency properties below:

$$\limsup_{\substack{(x',q',p',t')\to(x,q,p,t)\\ h\to 0}} S_h(x',q',p',t',\lambda;\Phi) \leq \mathcal{F}^*(x,q,p,t,\Phi,\nabla\Phi)$$

$$\liminf_{\substack{(x',q',p',t')\to(x,q,p,t)\\ h\to 0}} S_h(x',q',p',t',\lambda;\Phi) \geq \mathcal{F}_*(x,q,p,t,\Phi,\nabla\Phi)$$

Now, choose $\Phi \in C_b^\infty(\Omega)$ such that $\overline{u}-\Phi$ has a strict local maximum at $(x_0,q_0,p_0,t_0) \in \overline{\Omega}$ (without loss of generality assume $(\overline{u}-\Phi)(x_0,q_0,p_0,t_0)=0$).

First, suppose $p_0 > 0$. Then there exists a ball centered in (x_0,q_0,p_0,t_0) of radius $r > 0$ such that $\overline{u}(x,q,p,t) \leq \Phi(x,q,p,t),\ \forall (x,q,p,t) \in B((x_0,q_0,p_0,t_0),r) \subset \Omega$. Construct sequences $(x_\epsilon,q_\epsilon,p_\epsilon,t_\epsilon) \to (x_0,q_0,p_0,t_0)$ and $h_\epsilon \to 0$ as $\epsilon \to 0$ such that $u_{h_\epsilon}(x_\epsilon,q_\epsilon,p_\epsilon,t_\epsilon) \to \overline{u}(x_0,q_0,p_0,t_0)$ and $(x_\epsilon,q_\epsilon,p_\epsilon,t_\epsilon)$ is a maximum of $u_{h_\epsilon}-\Phi$ in $B((x_0,q_0,p_0,t_0),r)$. Denote $\zeta_\epsilon = (u_{h_\epsilon}-\Phi)(x_\epsilon,q_\epsilon,p_\epsilon,t_\epsilon)$. (Remark that $\zeta_\epsilon \to 0$ as $\epsilon \to 0$).

Then, $u_{h_\epsilon} \leq \Phi+\zeta_\epsilon$ inside the ball and since $S_h(x_\epsilon,q_\epsilon,p_\epsilon,t_\epsilon,u_{h_\epsilon}(x_\epsilon,q_\epsilon,p_\epsilon,t_\epsilon);u_{h_\epsilon}) = 0$, by the monotonicity property one obtains

$$S_h(x_\epsilon,q_\epsilon,p_\epsilon,t_\epsilon,\Phi(x_\epsilon,q_\epsilon,p_\epsilon,t_\epsilon)+\zeta_\epsilon;\Phi+\zeta_\epsilon) \leq 0.$$

Taking the limit (inf) $\epsilon \to 0$ together with the consistency of the scheme, one obtains the desired inequality

$$\mathcal{F}_*(x_0,q_0,p_0,t_0,\Phi,\nabla\Phi) \leq 0.$$

Suppose now that $p_0 = 0$. Construct sequences $(x_\epsilon,q_\epsilon,p_0,t_\epsilon) \to (x_0,q_0,p_0,t_0)$ and $h_\epsilon \to 0$ as $\epsilon \to 0$ such that $u_{h_\epsilon}(x_\epsilon,q_\epsilon,p_0,t_\epsilon) \to \overline{u}(x_0,q_0,p_0,t_0)$. Then,

$$\begin{aligned}\liminf_{\substack{x_\epsilon\to x_0\\ q_\epsilon\to q_0\\ t_\epsilon\to t_0\\ h_\epsilon\to 0}} S_h(x_\epsilon,q_\epsilon,p_0,t_\epsilon,\Phi(x_\epsilon,q_\epsilon,p_0,t_\epsilon),\Phi) &= \liminf_{\substack{x_\epsilon\to x_0\\ q_\epsilon\to q_0\\ t_\epsilon\to t_0\\ h_\epsilon\to 0}} (\Phi - M\Phi)(x_\epsilon,q_\epsilon,p_0,t_\epsilon)\\ &= (\Phi-(M\Phi)_*)(x_0,q_0,p_0,t_0).\end{aligned}$$

Since each u_{h_ϵ} is a solution of (4.3), using lemma 3.2 the above expression yields at the point (x_0,q_0,p_0,t_0):

$$\begin{aligned}0 &= (\Phi-(M\Phi)_*)(x_0,q_0,p_0,t_0)\\ &\geq (\Phi-(M\Phi)^*)(x_0,q_0,p_0,t_0)\\ &\geq (\Phi-(M^+\Phi))(x_0,q_0,p_0,t_0)\\ &= \mathcal{F}_*(x_0,q_0,p_0,t_0,\Phi,\nabla\Phi)\end{aligned}$$

achieving the desired inequality. □

4.2. Numerical Simulations. In this section, we give a simple example to illustrate the convergence result. For simplicity, we take here $d = 2$ and consider the numerical Hamiltonian $\mathcal{H}$ given by the monotone Local Lax-Friedrichs scheme [16] (where the two components of the gradient are explicit):

$$\mathcal{H}\left(t,x,q;a^+,a^-,b^+,b^-\right) = H\left(t,x,q;\frac{a^+ + a^-}{2},\frac{b^+ + b^-}{2}\right) - c_a\left(\frac{a^+ - a^-}{2}\right) - c_b\left(\frac{b^+ - b^-}{2}\right)$$

where $a^\pm = D_i^\pm v$, $b^\pm = D_j^\pm v$ and the constants c_a, c_b are defined as

$$c_a = \sup_{t,x,q,(r_1,r_2)} |\partial_{r_1} H(t,x,q;r_1,r_2)| \tag{4.5}$$

$$c_b = \sup_{t,x,q,(r_1,r_2)} |\partial_{r_2} H(t,x,q;r_1,r_2)|. \tag{4.6}$$

This particular discretization satisfies hypothesis (H4)-(H6) and therefore is in the setting of Proposition 4.1.

Taking v^n the approximate solution of v at t_n, the equation

$$S_h(x_I, q, p_k, t_{n+1}, v_{Ik}^{n+1}(q); v^n) = 0$$

allows an explicit expression of $v^{n+1}()$ as a function of past values $v^n()$:

$$v_{Ik}^{n+1}(q) = \begin{cases} \gamma_I \bigvee v_{Ik}^n(q) - \Delta t\left(\frac{v_{Ik}^n(q) - v_{Ik-1}^n(q)}{\Delta p} + \mathcal{H}(t_n, x_I, q, D^- v_{Ik}^n(q), D^+ v_{Ik}^n(q))\right) & \text{if } (I,q,k,n) \in G^\# \\ \min_{w \in W(x_I,q), k' \geq \delta/\Delta p} v_{Ik'}^n(g(w,q)) & \text{if } k = 0. \end{cases} \tag{4.7}$$

In order to illustrate the convergence of this particular numerical scheme, we present simulations using a simple hybrid vehicle model (see Section 2.1). Here the vehicle's energy state is a two-dimensional vector $y \in X = \mathbb{R}^2$, where $y = (y_1, y_2)$ denotes the state of charge of the battery and the fuel available in the fuel tank. These two quantities are the image of the remaining energy and have to be constrained to stay in the compact set $K = [0,1]^2$, where the energies quantities are normalized. The discrete variable $q \in Q = \{0,1\}$ is the ICE state, indicating whether the ICE is off ($q = 0$) or on ($q = 1$). The power output is a measurable function $u(\cdot) \in U$ where U is the admissible control set that is taken here as $U = [0, u_{\max}]$. We consider that the switch dynamics is given by $g(w,q) = |q - w|$.

Test 1. The energetic dynamical model is given by $f(t,x,u,q) = (-a_1 + qu, -q(a_2 + u))$, where $a_1, a_2 > 0$ are constant depletion rates of the battery's electric energy and the reservoir's fuel (whenever the ICE is on), respectively. The simulated instances use $a_1 = 0.10$, $a_2 = 0.15$ and $u_{\max} = 0.07$.

We test an instance where the set of initial positions $\mathcal{C}$ is the square centered at $(x_1, x_2) = (0.5, 0.5)$ and with half-lenght equal to 0.05. The computations are performed[1] for $(y_1, y_2) \in [-0.1, 1.1]^2$, $Q = 0, 1$, $p \in [0, T]$, where the final horizon is

[1] on 64-bit Mac OS using a 1.8GHz Intel Core i7 processor, with 4Gb of RAM

chosen as $T = 5.6$.

REMARK 4.1. *Considering the above dynamics, an exact* autonomy *of the system can be evaluated analytically. By autonomy, we mean the longest time during which the state remains inside K. Equivalently, the autonomy is the first time when the reachable set is empty: $T^* = \inf\{s \mid R_{\mathcal{C}}(s) = \emptyset\}$. Indeed, one can readily see that if no more admissible energy states are attainable after T^*, any admissible trajectory must come to a stop beyond this time. For this toy problem, given an initial conditions (x_1, x_2), the shortest time to empty the fuel reservoir is given by $t^* = x_1/(a_2 + u_{max})$. The charge left in the battery evaluated at this instant is given by $x_1(t^*) = x_1(0) - t^*(a_1 - u_{max})$. If $x_1(t^*) \leq 0$, it means the fuel cannot be consumed fast enough before the battery is depleted. This condition can be expressed in terms of the parameters of the model as $x_1(a_2 + u_{max}) \leq x_2(a_1 - u_{max})$. In this case, the autonomy is given by $T^* = x_1/(a_1 - u_{max})$. Otherwise, the autonomy is given by $T^* = (x_1 + u_{max}t^*)/a_1$.*

Simulations are running by fixing the CFL number to 0.9 in the condition (4.4) and then using several discretization steps $\Delta x_1 = \Delta x_2 = \Delta x$ and Δp. Since the exact value function is not known, the convergence error is analyzed with respect to a reference solution $V^{\#}$ computed with $\Delta x = 0.01$, $\Delta p = 0.028$. The L_∞ error, as a function of the discretization step sizes $\Delta x, \Delta p$, is then computed as being

$$e(\Delta x, \Delta p) = \sup_{(x,p,s,q) \in G^{\#}(\Delta x, \Delta p)} \mid v^{\#}(\Delta x, \Delta p) - V^{\#} \mid, \tag{4.8}$$

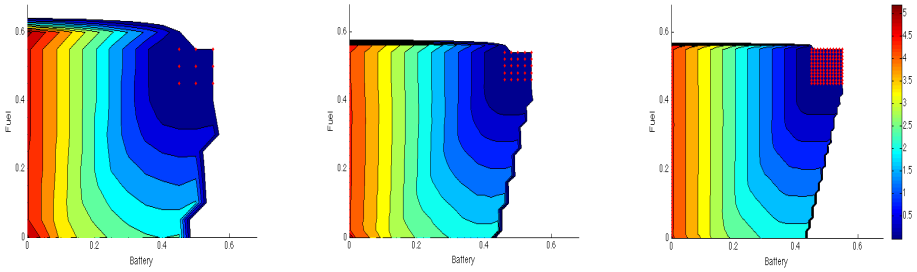
where $G^{\#}(\Delta x, \Delta p)$ is the numerical grid obtained using discretization steps $(\Delta x, \Delta p)$ and $v^{\#}(\Delta x, \Delta p)$ the corresponding numerical solution. Table 4.1 presents the L_∞-error and the total CPU running times as a function of Δx, Δp. In these simulations, δ is equal to 1. As can be seen from the numerical table, the error goes to 0 when both Δx and Δp tend to 0.

	$\Delta p = 0.0112$		$\Delta_p = 0.056$		$\Delta_p = 0.028$	
Δx	$e(\Delta x, \Delta p)$	CPU (s)	$e(\Delta x, \Delta p)$	CPU (s)	$e(\Delta x, \Delta p)$	CPU (s)
0.05	8.10e-2	7.8	7.23e-2	16.7	7.16e-2	50.1
0.04	5.462-2	10.8	4.61e-2	26.4	4.49e-2	77.7
0.03	3.17e-2	19.0	2.12e-2	47.1	1.74e-2	170.7
0.02	2.91e-2	45.0	1.50e-2	114.1	1.27e-2	451.2

TABLE 4.1
Convergence results and running times of an instance using $\delta = 1$.

Figure 4.1 shows, for different values of Δx and Δp, the minimum time values when the value function first crosses the 0-level set, i.e. $\mathcal{T}(x) = \min\{t \mid \exists q, p,\ v(x, q, p, t) \leq 0\}$. The function $\mathcal{T}$ represents for each x the first time the position is reached by an admissible trajectory starting from $\mathcal{C}$.

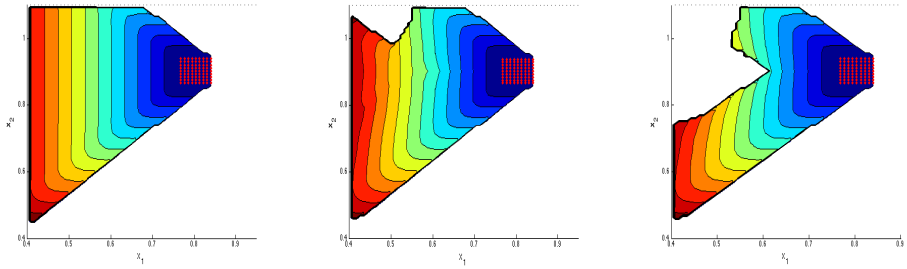
Test 2 The previous example aimed to show the convergence of the numerical method on a very simple model. Here we consider another model where the optimal trajectories may have more than one switch. The reachable set depends on the value of the lag variable δ. Set $f(t, x, u, q) = (-0.3, -0.3(2q - 1))$, and consider $\mathcal{C}$ the square centred at $(0.8, 0.9)^\intercal$ and with half lenght 0.04. The set of constraints K is $[-0.4, 1.1]^2$. Figure 4.2 shows the minimum time functions corresponding to different values of the lag δ. The simulations are performed on the computational domain



(a) $\Delta x = 0.05$, $\Delta p = 0.112$. (b) $\Delta x = 0.02$, $\Delta p = 0.56$. (c) $\Delta x = 0.01$, $\Delta p = 0.028$.

FIGURE 4.1. *Test 1. Contour plots of the minimum time function: Three discretization steps in Δx and Δp.*

$[-0.3, 1.2]^2 \times [0, T]$, with $T = 1.5$, $\Delta x_1 = \Delta x_2 = 0.009$, $\Delta p = 0.015$, and the CFL number is fixed to 0.6. In this example, the optimal strategies include several switches in order to keep the trajectories inside the set K. However, the number of possible switches depends on the value of δ. When δ is very small, the number of possible switches is large enough and the trajectories evolve in a large reachable set. If δ is large, then the number of possible switches decreases and the reachable set becomes smaller.



(a) $\delta = 0.02$. (b) $\delta = 0.4$. (c) $\delta x = 1$.

FIGURE 4.2. *Test 2. Contour plots of the minimum time functions corresponding to three different values of δ.*

Despite these illustrations of a convergent numerical scheme, we believe that a finer error analysis deserves further attention. This paper proofs the convergence of the numerical scheme, which is a first step in the numerical analysis of the SQVI. A more complete study should investigate more in detail the rate of convergence of numerical schemes, and the sensitivity analysis with respect to δ.

REFERENCES

[1] A. Arutyunov, V. Dykhta, and F. Lobo Pereira. Necessary conditions for impulsive nonlinear optimal control problems without a priori normality assumptions. *Journal of Optimization Theory and Applications*, 124:55–77(23), January 2005.

[2] M. Bardi and I. Capuzzo Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems and Control: Foundations and Applications. Birkhäuser, Boston, 1997.

[3] G. Barles, S. Dharmatti, and M. Ramaswamy. Unbounded viscosity solutions of hybrid control systems. *ESAIM: Control, Optimization and Calculus of Variations*, 16:176–193, 2010.

[4] G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Analysis*, 4(3):271–283, 1991.

[5] A. Bensoussan and J. L. Menaldi. Hybrid control and dynamic programming. *Dynamics of Continuous Discrete and Impulsive Systems*, (3):395–442, 1997.

[6] O. Bokanowski, N. Forcadel, and H. Zidani. Reachability and minimal times for state constrained nonlinear problems without any controllability assumption. *SIAM J. Control and Optimization*, 48(7):4292–4316, 2010.

[7] M. S. Branicky, V. S. Borkar, and S. K. Mitter. A unified framework for hybrid control: Model and optimal control theory. *IEEE Transactions On Automatic Control*, 43:31–45, 1998.

[8] B. Bruder and H. Pham. Impulse control problem on finite horizon with execution delay. *Stochastic Processes and their Applications*, 119:1436–1469, 2009.

[9] P. E. Caines, F. H. Clarke, X. Liu, and R.B. Vinter. A maximum principle for hybrid optimal control problems with pathwise state constraints. In *45th IEEE Conference on Decision and Control*, pages 4821 –4825, dec. 2006.

[10] M. G. Crandall, H. Ishii, and P-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S)*, (27), 1992.

[11] S. Dharmatti and M. Ramaswamy. Hybrid control systems and viscosity solutions. *SIAM J. Control Optim.*, 44(4):1259–1288, 2005.

[12] I. Capuzzo Dolcetta and L. C. Evans. Optimal switching for ordinary differential equations. *SIAM Journal On Control And Optimization*, 22(1):143–161, 1984.

[13] M. Garavello and B. Piccoli. Hybrid Necessary Principle. *SIAM Journal On Control And Optimization*, 43(5):1867–1887, 2005.

[14] O. Ley. Lower-bound gradient estimates for first-order Hamilton-Jacobi equations and applications to the regularity of propagating fronts. *Adv. Differential Equations*, 6(5):547–576, 2001.

[15] S. Osher and J.A. Sethian. Fronts propagating with curvature dependent speed: Algorithms based on hamilton-jacobi formulations. *Journal of Computational Physics*, 79:12–49, 1988.

[16] S. Osher and C-W. Shu. Higher order essentially non-oscillatory schemes for hamilton-jacobi equations. *SIAM Journal on Numerical Analysis*, 28(4):907–922, 1991.

[17] H.J. Sussmann. A maximum principle for hybrid optimal control problems. In *Proceedings of the 38th IEEE Conference on Decision and Control*, volume 1, pages 425 –430 vol.1, 1999.

[18] L. Yu, J.-P. Barbot, D. Boutat, and D. Benmerzouk. Observability normal forms for a class of switched systems with zeno phenomena. In *in Proceedings of the 2009 American Control Conference*, pages 1766–1771, 2009.

[19] H. Zhang and M. R. James. Optimal control of hybrid systems and a systems of quasi-variational inequalities. *SIAM Journal of Control Optimization*, 45:722–761, 2005.