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Probabilistic representation for solutions of a porous media type equation with Neumann boundary condition: the case of the half-line

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Summary: The purpose of this paper consists in proposing a generalized solution for a porous media type equation on a half-line with Neumann boundary condition and prove a probabilistic representation of this solution in terms of an associated microscopic diffusion. The main idea is to construct a stochastic differential equation with reflection which has a solution in law and whose marginal law densities provide the unique solution of the porous media type equation.

Key words: stochastic differential equations, reflection, porous media type equation, probabilistic representation.

2000 AMS-classification: 60H10, 60H30, 60H10, 60G46, 35C99.

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1 Introduction and preliminaries

In this work we focus on a porous media type equation given by

$$\begin{cases} \partial_t u(t,x) \in \frac{1}{2}\partial^2_{xx}\beta\left(u\right)(t,x) & : (t,x) \in (0,T] \times \mathbb{R}_+, \\ u(0,x) = u_0(x) & : x \in \mathbb{R}_+, \\ \partial_x\left(\beta\left(u\right)\right)(t,0) = 0, & : t \in (0,T]\,. \end{cases} \tag{1.1}$$

The natural analytical concept of (weak) solution of (1.1) is given in Definition 4 and it involves the restriction of the derivative of $\beta(u)$ on the boundary. We introduce here a new notion of solution that we call *generalized solution* for (1.1), which do not require, a priori, the existence of distributional derivatives for $\beta(u)$.

Under some minimal conditions, we will first concentrate on uniqueness of the generalized solutions (in a large class) and existence of a weak solution (smaller class). In particular, we will include the case when β is possibly discontinuous. Moreover we are interested in its probabilistic representation through the marginal laws of a stochastic process.

We formulate now some assumptions.

Assumption 1

i) $u_0 \in L^1\left(\mathbb{R}_+\right) \cap L^\infty\left(\mathbb{R}_+\right)$ is an initial probability density;

ii) $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing function with $\beta\left(0\right) = 0$.

iii) There is a constant c such that $\left|\beta\left(u\right)\right| \leq cu$.

With β we naturally associate a maximal monotone graph still denoted by the same letter $\beta : \mathbb{R}_+ \to 2^{\mathbb{R}_+}$, by filling the gaps, i.e. by identifying $\beta(x)$ with the interval $[\beta(x-), \beta(x+)]$. We consider now $\Phi : \mathbb{R}^*_+ \to \mathbb{R}_+$ such that $\beta(u) = \Phi^2(u)u, u \neq 0$. Again we associate naturally Φ with the non-negative graph (still denoted by Φ) $u \mapsto \sqrt{\frac{\beta(u)}{u}}$. Finally we extend the graph Φ it to $\mathbb{R}$, defining $\Phi(0) := [\liminf_{x\to 0+} \Phi(x), \limsup_{x\to 0+} \Phi(x)]$. Of course, if

β is continuous, the first line of (1.1) can be replaced by the most natural equality $\partial_t u(t,x) = \frac{1}{2}\partial^2_{xx}\beta(u)(t,x)$.

Definition 1. *i) We say that β is **non-degenerate** if there is a constant $c_0 > 0$ such that $\Phi \geq c_0$, for every $x > 0$.*

ii) We say that β is degenerate if $\lim_{u \to 0_+} \Phi(u) = 0$.

iii) We say that β is strictly increasing after some zero u_c if there is $u_c \geq 0$ such that $\beta\big|_{[0,u_c[} = 0$ and $\beta\big|_{[u_c,\infty[}$ is strictly increasing.

Often also the Assumption 2 below will be in force.

Assumption 2. We suppose that one of the following properties is verified.

i) β is non-degenerate.

ii) β is degenerate and there is a discrete ordered set of elements ($\{e_k\}$ of $\mathbb{R}_+ \cup \{+\infty\}$) so that u_0 has locally bounded variation on $)e_k, e_{k+1}($.

iii) β is strictly increasing after some zero u_c (in particular it is degenerate).

As we anticipated, the idea is to construct a stochastic process Y such that the (marginal) law of Y_t has a density given by $u(t,\cdot)$ for any $t \in [0,T]$. We look for Y as being a solution (in law) of the stochastic differential equation with reflection

$$\begin{cases} dX_t = \Phi(u(t,X_t))\, dB_t + dK_t, \\ K \text{ increasing process } \int_0^T Y_s dK_s = 0, \\ u(s,.) = \text{law density of } Y_t, \end{cases} \tag{1.2}$$

which has a weak solution X whose law density is the unique solution of the (1.1).

As far as our knowledge is concerned, this paper could be the first one studying the probabilistic representation of a non-linear partial differential equation (PDE) with Neumann boundary conditions on some domain, through a reflected non-linear diffusion.

The problem of probabilistic representation related to a PDE, related to a class $\mathcal{A}$ of solutions, is the following. For each of $u \in \mathcal{A}$, there exists a stochastic process, solving some form of stochastic differential equation whose coefficients involve the law of the process (McKean-Vlasov type), whose marginal laws are given by u. Solutions of those stochastic differential equations are also called non-linear diffusions. The PDE is intended as a non-linear forward Kolmogorov's equation corresponding to the non-linear diffusion.

The paper provides a bridge between two big areas of stochastic analysis: stochastic differential equations with reflection on some domain, non-linear diffusions on the whole line. As we will see the literature is rich of contributions in both topics, but in principle no one connects them.

1. **Non-linear diffusion problems**

   There are several contributions to the study of equations stated in the first line of (1.1) but on the whole line or even on $\mathbb{R}^d$. That equation, which will be precisely stated in (3.7), was first investigated by [9] for existence, [13] for uniqueness and [10] for continuous dependence on coefficients.

   The physical interpretation of the probabilistic representation is the following. The singular non-linear diffusion equation (3.7) describes a *macroscopic* phenomenon for which the probabilistic representation tries to give a *microscopic* probabilistic interpretation via a non-linear stochastic diffusion equation modeling the evolution of a single point on the layer.

   To our knowledge, the first author who considered a probabilistic representation for the solution of a non-linear deterministic PDE (on the whole line), was McKean [22], especially in relation to the so called propagation of chaos. He supposed to have smooth coefficients. After that, the literature grew and nowadays there is a vast amount of contributions to the subject, particularly when the non-linearity $\beta(u)$ appears inside the first order part, as e.g. in Burgers equations (see for instance the survey papers [18] and [28]).

   A probabilistic interpretation of (3.7) when $\beta(u) = |u|u^{m-1}$, for $m > 1$

was provided for instance in [8] in which probabilistic representations of the Barenblatt solutions and of a large class of solutions were given. Later, when Φ is of class C^3, Lipschitz, β is non-degenerate and u_0 is smooth enough, [19] provided also strong solutions to the probabilistic representation problem, see more precisely Remark 17. In particular, the probabilistic representation of the porous media on $\mathbb{R}$ was studied in the case of irregular coefficients in [11, 4] with refinements in [5, 7, 6]. In particular [11] represented when β is non-degenerated all the solutions in the sense of distributions under Assumption 1. Moreover also the uniqueness of the corresponding non-linear diffusions was established. When β is degenerate, under Assumptions 1. and 2. ii) or 2. iii), [4] has provided again probabilistic representations, but not uniqueness of those. Some improvements also appeared in [7, 6], at least when Φ is continuous. [7] provides probabilistic representation of the Barenblatt solutions when $\beta(u) = u^m, \frac{3}{5} < m < 1$, i.e. in the case of fast diffusions.

2. **Stochastic differential equations with reflection**. There is a vast literature in the subject, in the one-dimensional case, and in the multidimensional case as well. It is for us impossible to quote all those. In the half-line case $\mathbb{R}_+$, such an equation can be formulated as follows:

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt + dK_t,$$

where $\sigma, b : [0, T] \times \mathbb{R}$ are Borel functions and K is an increasing process such that $\int_0^T X_s dK_s = 0$. The solution is the couple (X, K). When σ and b are Lipschitz the theory is well-developed, see for instance [21], [15] and [25] (and references therein). All the three papers have treated the case when the coefficients are time-independent. Elements related to the time-dependent case appear for instance in [16] and in [24], chapter IX, Exercise (2.14), page 385.
When the coefficients are non-Lipschitz, but are Lipschitz with a logarithmic correction and the domain is a half-line [12], proved recently the existence of a strong solution.

As far as weak (in law) solutions of SDEs with reflection, the pioneering work is [27], which solves some submartingale problem related to

stochastic differential equations with reflection under relatively general conditions on the (even time-dependent) coefficients. An interesting work is also [23] which constructed solutions of symmetric (time-homogeneous) diffusions with reflection on a domain of $\mathbb{R}^n$, via Dirichlet forms.

Reflected diffusions are naturally candidates for the probabilistic representation of solutions of (linear) PDE with Neumann boundary condition in the following sense. Given a specific solution u to a Fokker-Planck type PDE, with Neumann boundary conditions, a process X represents it probabilistically if the marginal laws of X are solutions of the PDE. As far as we know, even in this linear case, the point is not clear in the literature. For instance, by applying some Itô formula type, it is possible to show that solutions of SDEs (or martingale problems) solve a PDE with boundary condition, in some sense. However to show that a given solution to a Fokker-Planck PDE can be represented through a process is rarely explained. This is related to the study of uniqueness of the mentioned PDE, see for instance [11], Theorem 3.8, for an equation in the whole line.
We conclude this discussion about probabilistic representation of linear PDEs mentioning [17], which gives a representation of the solution of an elliptic problem with unbounded monotone drift in term of the invariant measure of a reflection diffusion equation considered by [15].

We are aware that the Neumann problem on the half-line constitutes somehow a toy model, however at our knowledge, there are no results in the literature about well-statement of that problem. If x varies in a bounded domain (for instance a compact interval), there are some contributions at least in the case when $\beta(u) = u^m, 0 < m < 1$, which constitutes the case of the classical porous media equation. Given an integrable initial condition u_0, [1], through the techniques of maximal accretivity, see e.g. [26], [3], a C^0-type solution (or mild solution), see Chapter IV.8 of [26]. The technique consists in showing that the elliptic corresponding operator is m-accretive, whose step was performed by [14]. In Corollary 3.5, [1] shows that (when $u_0 \in L^\infty$), that the solution is even classical and it is therefore a weak so-

lution in the sense of Definition 4, adapted to the case when an interval replaces the real line.

The paper is organized as follows. After the introduction above, at Section 2 we introduce the basic definition of solutions and some notations. At Section 3, we discuss existence and uniqueness of (1.1) and we remark that the solutions have some minimal regularity properties. Section 4 is devoted to the existence of the probabilistic representation in the form of a solution to a non-linear (in the sense that the marginal densities appear in the coefficients) stochastic differential equation, with reflection. The Appendix is devoted to the equivalence of weak and generalized solutions under some minimal regularity conditions.

2 Preliminaries

Let I be a real interval. Given a function $\varphi : [0,T] \times I \to \mathbb{R}$, $(t,x) \mapsto \varphi(t,x)$, we denote (if it exists), by φ' (resp. φ'') the partial derivative $\partial_x \varphi$ (resp. second partial derivative $\partial^2_{xx}\varphi$) with respect to the second argument, defined again on $[0,T] \times I$. If I is closed, then the derivatives are defined as continuous extensions from $[0,T] \times \mathrm{Int} I$.

In this paper $C_0^\infty(\mathbb{R}_+)$ will denote the set of functions $\varphi : \mathbb{R}_+ \to \mathbb{R}$ which are restrictions of smooth functions with compact support defined on $\mathbb{R}$. We denote by $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ the space of absolutely continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that for every compact subset K of $\mathbb{R}_+$ $\int_K |f'(x)|dx$ is finite. Of course for any $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ and every compact K of $\mathbb{R}$ we have $\int_K |f(x)|dx < \infty$.

If $I = \mathbb{R}$ or $\mathbb{R}_+$, we denote a bit abusively by $W^{1,1}_{\mathrm{loc}}([0,T] \times I)$ the set of $f : [0,T] \times \mathbb{R} \to I$ such that for almost all t, we have $f(t,\cdot) \in W^{1,1}_{\mathrm{loc}}(I)$ and for $\int_{[0,T]\times K} |\partial_x f(t,x)| < \infty$, for every compact K of I.

Definition 2. *A function $u \in L^1([0,T] \times \mathbb{R}_+)$ is called **generalized solution** for equation (1.1) if, for any $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi'(0) = 0$, we*

have

$$\int_0^\infty \varphi(x)\, u(t,x)\, dx = \int_0^\infty \varphi(x)\, u_0(x)\, dx + \frac{1}{2}\int_0^t \int_0^\infty \varphi''(x)\, \eta_u(s,x)\, dxds, \tag{2.3}$$

where $\eta_u : [0,T]\times\mathbb{R}_+ \to \mathbb{R}$, $\eta_u \in L^1([0,T]\times\mathbb{R}_+)$ *is such that*

$$\eta_u(t,x) \in \beta(u(t,x)), \quad dt\otimes dx - a.e.\ (t,x)\in[0,T]\times\mathbb{R}_+.$$

We remark a generalized solution in the same spirit, for porous media equations but with Dirichlet boundary conditions was given in [[3] page 226]). Moreover, in our case the test function is time independent.

Remark 3. *The formal justification of previous formula comes out from the following observation. Suppose that* (u,η_u) *is a smooth solution to (1.1) and* u_0 *is continuous. Then, for every* $x\in\mathbb{R}_+$, *we have*

$$u(t,x) = u_0(x) + \frac{1}{2}\int_0^t \eta_u''(s,x)\, ds.$$

Let φ *be a test function from* $C_0^\infty(\mathbb{R}_+)$ *such that* $\varphi'(0)=0$. *In this particular situation we have*

$$\begin{aligned}
\int_0^\infty \varphi(x)\, u(t,x)\, dx &= \int_0^\infty \varphi(x)\, u_0(x)\, dx + \frac{1}{2}\int_0^t\int_0^\infty \varphi(x)\, \eta_u''(s,x)\, dxds \\
&= \int_0^\infty \varphi(x)\, u_0(x)\, dx + \frac{1}{2}\int_0^t \varphi(0)\, \eta_u'(s,0)\, ds \\
&\qquad -\frac{1}{2}\int_0^t\int_0^\infty \varphi'(x)\, \eta_u'(s,x)\, dxds \\
&= \int_0^\infty \varphi(x)\, u_0(x)\, dx + \frac{1}{2}\int_0^t \varphi(0)\, \eta_u'(s,0)\, ds \\
&\quad +\frac{1}{2}\int_0^t\int_0^\infty \varphi''(x)\, \eta_u(s,x)\, dxds - \frac{1}{2}\underbrace{\int_0^t \varphi'(0)\, \eta_u(s,0)\, ds}_{=0},
\end{aligned}$$

which constitutes indeed (2.3).

As we mentioned in the introduction, the natural analytical concept of **solution** should involve the first spatial derivative at the boundary.

We consider a couple (u, η_u) satisfying Definition 2 and suppose that $\eta_u \in W^{1,1}_{loc}([0,T]\times\mathbb{R}_+)$ in agreement with the notations of Section 2. In particular for every compact $K \in \mathbb{R}$ we have

$$\int_{[0,T]\times K} \left|\eta'_u(s,x)\right| dsdx < \infty. \tag{2.4}$$

Definition 4. *The couple (u, η_u) is said to be a weak solution of (1.1) if for every $\varphi \in C_0^\infty(\mathbb{R}_+)$ we have*

$$\int_0^\infty \varphi(x)\, u(t,x)\, dx = \int_0^\infty \varphi(x)\, u_0(x)\, dx - \int_0^t \int_0^\infty \varphi'(x)\, \eta'_u(s,x)\, dxds. \tag{2.5}$$

Proposition 5. *Let (u, η_u) be a couple such that $\eta_u(t,x) \in \beta(u(t,x))$ and $\eta_u \in W^{1,1}_{loc}([0,T]\times\mathbb{R}_+)$. Then u is a generalized solution if and only if it is a weak solution in the sense of Definition 4.*

Proof. See Appendix. □

3 The porous media equation on half-line with Neumann boundary condition

In this part of the paper we will study existence and uniqueness of the generalized solution for equation (1.1). We also show the connection with the notion of weak solutions.

This will be done using the known results on the whole line $\mathbb{R}$. To this purpose, we start extending the initial condition to the real line by the following construction.
Let $\overline{u_0} \in (L^1 \cap L^\infty)(\mathbb{R})$ be defined by

$$\overline{u_0}(x) = \begin{cases} \frac{1}{2}u_0(x) & , x \geq 0 \\ \frac{1}{2}u_0(-x) & , x < 0. \end{cases} \tag{3.6}$$

and $\overline{\beta} : \mathbb{R} \to \mathbb{R}$ by

$$\overline{\beta}(u) = \frac{1}{2}\beta(2u), \quad u \in \mathbb{R}.$$

We can now consider the corresponding porous media equation on the whole line

$$\begin{cases} \partial_t \overline{u}(t,x) = \frac{1}{2}\partial^2_{xx}\overline{\beta}\left(\overline{u}\right)(t,x) & : (t,x) \in (0,T]\times\mathbb{R}, \\ \overline{u}(0,.) = \overline{u}_0, \end{cases} \tag{3.7}$$

which, by Proposition 3.4 from [11] (see also [10]) has a unique solution in the sense of distributions, i.e. there exists a unique couple $(\overline{u}, \eta_{\bar{u}}) \in \left(L^1 \cap L^\infty\right)([0,T]\times\mathbb{R})$ such that

$$\int_{\mathbb{R}} \varphi(x)\,\overline{u}(t,x)\,dx = \int_{\mathbb{R}} \varphi(x)\,\overline{u}_0(x)\,dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \varphi''(x)\,\eta_{\bar{u}}(s,x)\,dxds, \tag{3.8}$$

for all $\varphi \in C_0^\infty(\mathbb{R})$ and

$$\eta_{\bar{u}}(t,x) \in \overline{\beta}\left(\overline{u}(t,x)\right), \text{ for } dt\otimes dx - \text{ a.e. } (t,x) \in [0,T]\times\mathbb{R}.$$

Remark 6. *Since $\overline{u_0}$ is even, the solution $\overline{u}$ to equation (3.7) and $\eta_{\bar{u}}$ are also even. In fact, by applying* (3.8) *to $x \mapsto \varphi(-x)$ and making then the change of variables $x \mapsto -x$, we show that $(t,x) \mapsto \overline{u}(t,-x)$ is also a solution. The result follows by uniqueness of* (3.7).

Proposition 7. *We define $v, \eta_v : [0,T]\times\mathbb{R}_+$, setting $v(t,x) = 2\overline{u}(t,x)$ and $\eta_v(t,x) = 2\eta_{\bar{u}}(t,x), \forall (t,x) \in [0,T]\times\mathbb{R}_+$. The couple (v,η_v) is a generalized solution to equation (1.1) in the sense of Definition 2.*

Proof. Since

$$2\overline{\beta}\left(\overline{u}\right) = \beta\left(2\overline{u}\right) = \beta\left(v\right),$$

we first observe that $\eta_v(t,x) \in \beta(v(t,x))dtdx$ a.e. Moreover η_v belongs to $L^1([0,T]\times\mathbb{R}_+)$ since $\eta_{\bar{u}}$ belongs to $L^1([0,T]\times\mathbb{R})$. It remains to prove that (v,η_v) satisfies (2.3) for all $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi'(0) = 0$.

Step I

First we prove that (2.3) is true for all test functions $\varphi \in C_0^\infty(\mathbb{R}_+)$ with $\varphi'(0) = 0$ and which extend to an even smooth function $\bar{\varphi}$ with compact support on $\mathbb{R}$. By (3.8) we have

$$\int_{\mathbb{R}} \overline{\varphi}(x)\,\overline{u}(t,x)\,dx = \int_{\mathbb{R}} \overline{\varphi}(x)\,\overline{u}_0(x)\,dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \overline{\varphi}''(x)\,\eta_{\bar{u}}(s,x)\,dxds.$$

Since $\overline{\varphi}$, $\overline{u}_0$ and $\overline{u}(t,\cdot)$, for any $t \in [0,T]$ are even functions, we get, for the first two terms, that

$$\int_{\mathbb{R}} \overline{\varphi}(x)\,\overline{u}(t,x)\,dx = \int_0^\infty \varphi(x)\,2\overline{u}(t,x)\,dx = \int_0^\infty \varphi(x)\,v(t,x)\,dx,$$

$$\int_{\mathbb{R}} \overline{\varphi}(x)\,\overline{u}_0(x)\,dx = \int_0^\infty \overline{\varphi}(x)\,2\overline{u}_0(x)\,dx = \int_0^\infty \overline{\varphi}(x)\,u_0(x)\,dx.$$

Since $\eta_v = 2\eta_{\bar{u}}$ we obtain

$$\begin{aligned}\int_{\mathbb{R}} \bar{\varphi}''(x)\,\eta_{\bar{u}}(s,x)\,dx &= \int_0^\infty \overline{\varphi}''(x)\,2\eta_{\bar{u}}(s,x)\,dx \\ &= \int_0^\infty \varphi''(x)\,\eta_v(s,x)\,dx,\ s \in [0,T].\end{aligned}$$

This proves (1.1) for the restricted class of φ which extend to an even smooth function on $\mathbb{R}$ with compact support.

Step II

Now we can prove the general statement.

Let $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi'(0) = 0$. We extend it to an even function

$$\overline{\varphi}(x) = \begin{cases} \varphi(x) & : x \geq 0 \\ \varphi(-x) & : x < 0, \end{cases}$$

which has compact support, but it does not belong necessarily to $C_0^\infty(\mathbb{R})$. In order to have a proper test function for evaluating it in (3.8), we need to convolute it with a mollifier.

Let $\rho \in C_0^\infty(\mathbb{R})$ be such that $\rho \geq 0$ for $|x| \geq 1$, $\rho(x) = \rho(-x)$ and $\int_{\mathbb{R}} \rho(x)\,dx = 1$. For an example of such function see e.g. [2].

We set $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right)$ as a mollifier and we take the regularization

$$\begin{aligned}\overline{\varphi}_\varepsilon(x) &= \int_{\mathbb{R}} \overline{\varphi}(x)\,\rho_\varepsilon(x-y)\,dy \\ &= \int_{\mathbb{R}} \overline{\varphi}(x-\varepsilon y)\,\rho(y)\,dy, \quad \forall x \in \mathbb{R}.\end{aligned}$$

It is well-known that $\overline{\varphi}_\varepsilon \in C_0^\infty(\mathbb{R})$ and we can check that $\overline{\varphi}_\varepsilon$ is also even.

Indeed we have

$$\begin{aligned}\overline{\varphi}_\varepsilon(-x) &= \int_{\mathbb{R}} \overline{\varphi}(-x-\varepsilon y)\,\rho(y)\,dy \\ &= \int_{\mathbb{R}} \overline{\varphi}(x+\varepsilon y)\,\rho(y)\,dy \\ &= \int_{\mathbb{R}} \overline{\varphi}(x-\varepsilon y)\,\rho(-y)\,dy = \overline{\varphi}_\varepsilon(x).\end{aligned}$$

By *Step I* we get

$$\int_0^\infty \overline{\varphi}_\varepsilon(x)\,v(t,x)\,dx = \int_0^\infty \overline{\varphi}_\varepsilon(x)\,u_0(x)\,dx + \frac{1}{2}\int_0^t\int_0^\infty (\overline{\varphi}_\varepsilon)''(x)\,\eta_v(s,x)\,dxds.$$

Since $\overline{\varphi}_\varepsilon\big|_{\mathbb{R}_+} \in C_0^\infty(\mathbb{R}_+)$ we have

$$(\overline{\varphi}_\varepsilon)''(x) = \int_{\mathbb{R}} \overline{\varphi}''(x-\varepsilon y)\,\rho(y)\,dy = \left(\overline{\varphi}''\right)_\varepsilon(x), \quad \forall x \in \mathbb{R}_+;$$

then we can pass to the limit for $\varepsilon \to 0$ and conclude the proof. □

Corollary 8. *Under Assumptions 1. and 2., the generalized solution of* (1.1) *is also a weak solution.*

Proof. Let $(\bar{u}, \eta_{\bar{u}})$ be the solution of (3.7), i.e. on the real line with initial condition $\bar{u}_0$ as in (3.6). Under Assumptions 1. and 2., Proposition 4.5 a) of [4] says that for a.e. $t \in [0,T]$, $\eta_u(t,\cdot) \in H^1(\mathbb{R})$ and $\int_{[0,T]\times\mathbb{R}} \eta_u(t,x)^2 dtdx < \infty$. This implies that for every compact real interval K, $\eta'_{\bar{u}} \in L^1([0,T]\times K)$. In particular $\eta_{\bar{u}} \in W^{1,1}_{\mathrm{loc}}([0,T]\times\mathbb{R})$. By the proof of Proposition 9, the solution (u,η_u) equals the restriction of $(2\bar{u}, 2\bar{\eta})$ to $\mathbb{R}_+$. This shows that $\eta_u)$ belongs to $W^{1,1}_{\mathrm{loc}}([0,T]\times\mathbb{R}_+)$. By Proposition 5, u is also a weak solution. □

Proposition 9. *Equation (1.1) has a unique generalized solution in the sense of Definition 2.*

Proof. Existence has been the object of Proposition 7, so we proceed now to uniqueness. Let (v,η_v) be a generalized solution of (1.1), i.e. for any $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi'(0) = 0$ we have

$$\int_0^\infty \varphi(x)\,v(t,x)\,dx = \int_0^\infty \varphi(x)\,u_0(x)\,dx + \frac{1}{2}\int_0^t\int_0^\infty \varphi''(x)\,\eta_v(s,x)\,dxds. \tag{3.9}$$

We define $\bar{u}_0 : \mathbb{R} \to \mathbb{R}$ by

$$\overline{u_0}(x) = \begin{cases} \frac{1}{2}u_0(x) & , x \geq 0 \\ \frac{1}{2}u_0(-x) & , x < 0 \end{cases}$$

We extend the considered solution to $[0,T] \times \mathbb{R}$ setting

$$\overline{u}(t,x) = \begin{cases} \frac{1}{2}v(t,x) & , x \geq 0 \\ \frac{1}{2}v(t,-x) & , x < 0, \end{cases}$$

$$\eta_{\bar{u}}(t,x) = \begin{cases} \frac{1}{2}\eta_v(t,x) & , x \geq 0 \\ \frac{1}{2}\eta_v(t,-x) & , x < 0. \end{cases}$$

Obviously we have

$$\eta_{\bar{u}}(t,x) \in \frac{1}{2}\beta(2\overline{u}(t,x)) = \bar{\beta}(\bar{u}(t,x)), \quad dt \otimes dx - a.e.\ (t,x) \in [0,T] \times \mathbb{R}_+.$$

We aim now at showing that $(\overline{u}, \overline{\eta_u})$ constructed above, is a solution to equation (3.7) in order to use the uniqueness for the equation on the whole line. To this purpose we have to prove that

$$\int_{\mathbb{R}} \varphi(x)\,\overline{u}(t,x)\,dx = \int_{\mathbb{R}} \varphi(x)\,\overline{u}_0(x)\,dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \varphi''(x)\,\eta_{\bar{u}}(s,x)\,dxds,$$

for all $\varphi \in C_0^\infty(\mathbb{R})$.

Let us now fix $\varphi \in C_0^\infty(\mathbb{R})$ and we can define $\psi : \mathbb{R}_+ \to \mathbb{R}$ by $\psi(x) = \varphi(x) + \varphi(-x)$. Obviously $\psi \in C_0^\infty(\mathbb{R}_+)$ and $\psi'(0) = 0$, so it is a test function for (3.9). This gives

$$\int_0^\infty \psi(x)\,v(t,x)\,dx = \int_0^\infty \psi(x)\,u_0(x)\,dx + \frac{1}{2}\int_0^t \int_0^\infty \psi''(x)\,\eta_v(s,x)\,dxds.$$

The left-hand side above yields

$$\begin{aligned} \int_0^\infty \psi(x)\,v(t,x)\,dx &= \int_0^\infty (\varphi(x) + \varphi(-x))\,v(t,x)\,dx \\ &= \int_0^\infty \varphi(x)\,v(t,x)\,dx + \int_{-\infty}^0 \varphi(x)\,v(t,-x)\,dx \\ &= \int_{\mathbb{R}} \varphi(x)\,2\overline{u}(t,x)\,dx, \end{aligned}$$

by the obvious change of variable $x \mapsto -x$. The same technique shows that

$$\int_0^\infty \psi(x)\,u_0(x)\,dx = \int_{\mathbb{R}} \varphi(x)\,2\overline{u}_0(x)\,dx,$$

and also

$$\int_0^\infty \psi''(x)\,\eta_v(s,x)\,dx = 2\int_{\mathbb{R}} \varphi''(x)\,\eta_{\bar{u}}(s,x)\,dx, \quad \text{for all } s\in[0,T].$$

This implies

$$2\int_{\mathbb{R}} \varphi(x)\,\overline{u}(t,x)\,dx = 2\int_{\mathbb{R}} \varphi(x)\,\overline{u_0}(x)\,dx + 2\int_0^t\int_{\mathbb{R}} \varphi''(x)\,\eta_{\bar{u}}(s,x)\,dxds,$$

where

$$\eta_{\bar{u}}(t,x)\in\bar{\beta}\left(\overline{u}(t,x)\right), \quad a.e.\ (t,x)\in[0,T]\times\mathbb{R}_+$$

and we get that

$$\begin{cases} \partial_t\overline{u}(t,x)\in\frac{1}{2}\partial_{xx}^2\left(\bar{\beta}\left(\overline{u}(t,x)\right)\right) & :(t,x)\in(0,T]\times\mathbb{R}, \\ \overline{u}(0,x)=\overline{u_0}(x). \end{cases}$$

Finally, this shows that $\overline{u}$ is a solution to equation (3.7). As said at the beginning of the proof, the uniqueness for v follows from uniqueness of (3.7). □

Remark 10. *By Remark 1.6 of [4], the solution $\bar{u}$ of (3.7) on the real line, has mass conservation and it is always non-negative. Taking into account the construction of the generalized solution, we also get a similar result for the generalized solution v of (1.1). This means the following.*

1. *$v\geq 0$ a.e.*
2. *$\int_{\mathbb{R}} v(t,x)dx = 1,\ \forall t\in[0,1]$.*

This explains why the values of β on $\mathbb{R}_-$ are not important, see Assumptions 1. and 2.

4 The basic construction of the linear reflected diffusion

In this section we are interested in the probabilistic representation of equation (1.1), under Assumptions 1. and 2. from the introduction. More precisely we aim at characterizing *all* the solutions of (1.1).

In this case we remind that we can write $\beta(u) = \Phi^2(u)\, u$, for Φ being a non-negative graph generated by a bounded function Φ.

We introduce now $\overline{\Phi} : \mathbb{R} \to \mathbb{R}$ by $\overline{\Phi}(u) = \Phi(2u)$. With this notation we have $\overline{\beta}(u) = u\, \overline{\Phi}^2(u)$. Let $\bar{u}_0$ be as in (3.6).

According to Theorem 4.4 in [11], Theorem 5.4 in [4] taking into account the proof of Theorem 2.6 in [5], equation (3.7) admits a probabilistic representation, in the sense that there is a solution Y (in law) of

$$\begin{cases} Y_t \in Y_0 + \int_0^t \bar{\Phi}\left(\overline{u}(s, Y_s)\right) dW_s, t \in [0,T], \\ \overline{u}(t,.) = \text{law density of } Y_t, t \in [0,T], \\ \bar{u}(0,\cdot) = \bar{u}_0, \end{cases} \tag{4.10}$$

for some Wiener process (W_s) on some probability space.

The precise meaning of the first line of (4.10) is the following:

$$Y_t \in Y_0 + \int_0^t \chi_{\overline{u}}(s, Y_s)\, dW_s, t \in [0,T],\ \chi_{\bar{u}}(t,y) \in \bar{\Phi}(\bar{u}(t,y)), dtdy \text{ a.e.} \tag{4.11}$$

In order not to loose the reader, without restriction of generality, we suppose in this section that Φ (and therefore $\bar{\Phi}$) is continuous on $\mathbb{R}_+$ so that we can write $\chi_u = \Phi(u)$.

Remark 11. *We remind that we have defined Φ (and therefore $\bar{\Phi}$) artificially at zero in the Introduction. Observe that this has no influence in the probabilistic representation since $\int_0^T 1_{\{\bar{u}(s,Y_s)=0\}} \bar{\Phi}^2(\bar{u}(s,Y_s))ds = 0$ a.s. since its expectation gives $\int_0^T ds \int_{\mathbb{R}} 1_{\{\bar{u}(s,y)=0\}} \bar{u}(s,y)\bar{\Phi}^2(\bar{u}(s,y))dy = 0$.*

Our purpose is to use this result in order to get a corresponding one for the porous media equation on the half-line with Neumann boundary condition. We start with a preliminary result.

Lemma 12. *Let $(Y, \bar{u})$ be the solution of (4.10). Let us denote by $\nu_t, t \in [0,T]$, the marginal laws of $X_t = |Y_t|, t \in [0,T]$. Then for all $t \in [0,T]$, ν_t has a density which is given by $v(t,\cdot)$ where $v = 2\, \overline{u}|_{[0,T]\times\mathbb{R}_+}$.*

Proof. We remind that $\overline{u}$ is the law density of Y_t from equation (4.10). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a bounded Borel function and $t \in [0,T]$. We notice that

$$\begin{aligned}\mathbb{E}\left(\varphi\left(X_t\right)\right) &= \mathbb{E}\left(\varphi\left(|Y_t|\right)\right) = \int_{\mathbb{R}} \varphi\left(|y|\right) \overline{u}\left(t,y\right) dy = \\ &= \int_{-\infty}^{0} \varphi\left(-y\right) \overline{u}\left(t,y\right) dy + \int_{0}^{\infty} \varphi\left(y\right) \overline{u}\left(t,y\right) dy \\ &= \int_{0}^{\infty} \varphi\left(y\right) 2\ \overline{u}\left(t,y\right) dy;\end{aligned}$$

since $\bar{u}(t,\cdot)$ is even, this concludes the proof.

□

Let v be as in Lemma 12. We continue by investigating the stochastic equation solved by $X_t = |Y_t|, t \in [0,T]$ where Y is the solution of (4.10).

By Itô-Tanaka formula (see e.g. [24] Theorem (1.2) Chapter VI) we get

$$X_t = |Y_t| = |Y_0| + \int_0^t sgn\left(Y_s\right) \bar{\Phi}\left(\overline{u}\left(s,Y_s\right)\right) dW_s + L_t^Y\left(0\right), \tag{4.12}$$

where

$$sgn(x) = \begin{cases} 1 & , x > 0 \\ -1 & , x < 0 \\ 0 & , x = 0 \end{cases}$$

and $(L_t^Y(0))$ is the local time of the semimartingale Y at zero. Since Y is a local martingale, this is characterized by

$$L_t^Y\left(0\right) = \frac{1}{2} \lim_{\varepsilon \to 0} \int_0^t \frac{1}{\varepsilon} 1_{\{Y_s \in (-\varepsilon,\varepsilon)\}} d\left\langle Y\right\rangle_s, \tag{4.13}$$

see e.g. Corollary (1.9) Chapter 6 [24]. We define

$$B_t^1 = \int_0^t sgn\left(Y_s\right) dW_s,$$

so that

$$\left\langle B^1\right\rangle_t = \int_0^t \left(sgn\right)^2\left(Y_s\right) ds = \int_0^t 1_{\{Y_s \neq 0\}} ds.$$

Possibly enlarging the probability space, let B^2 be an independent Brownian motion of W. We set

$$B := B^1 + \int_0^{\cdot} 1_{\{Y_s = 0\}} dB_s^2. \tag{4.14}$$

Since $\langle W, B^2\rangle_t = 0$, it follows that

$$\langle B\rangle_t = \int_0^t 1_{\{Y_s\neq 0\}} ds + \int_0^t 1_{\{Y_s=0\}} ds = t, \forall t \in [0,T].$$

By Lévy characterization theorem of the Brownian motion, it follows that B is a Wiener process.

By (4.12), since L^Y is a bounded variation process, we show that

$$\langle X\rangle_t = \int_0^t 1_{\{Y_s\neq 0\}} \overline{\Phi}^2(\bar{u}(s,Y_s))ds = \int_0^t 1_{\{Y_s\neq 0\}} \Phi^2(2\bar{u}(s,Y_s))ds \qquad (4.15)$$

$$= \int_0^t 1_{\{X_s\neq 0\}} \Phi^2(v(s,X_s))ds,$$

because $\bar{u}(s,y) = \dfrac{v}{2}(s,|y|),\ s\in[0,T],\ y\in\mathbb{R}$. Coming back to (4.13), again by Corollary (1.9) Chapter VI of [24] and (4.10), we get

$$\begin{aligned}
L_t^Y(0) &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t 1_{\{Y_s\in(-\varepsilon,\varepsilon)\}} d\langle Y\rangle_s \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t 1_{\{Y_s\in(-\varepsilon,\varepsilon)\}} \overline{\Phi}^2(\overline{u}(s,Y_s))\,ds \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t 1_{\{|Y_s|\in[0,\varepsilon)\}} \Phi^2(2\overline{u}(s,Y_s))\,ds \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t 1_{\{X_s\in[0,\varepsilon)\}} \Phi^2(v(s,X_s))\,ds \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t 1_{\{X_s\in(0,\varepsilon)\}} \Phi^2(v(s,X_s))\,ds \\
&\qquad + \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t 1_{\{X_s=0\}} \Phi^2(v(s,X_s))\,ds.
\end{aligned}$$

because of Lemla 12. By (4.15) it follows that

$$\begin{aligned}
L_t^Y(0) &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t 1_{\{X_s\in[0,\varepsilon)\}} d\langle X\rangle_s \\
&\qquad + \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t 1_{\{X_s=0\}} \Phi^2(v(s,X_s))\,ds.
\end{aligned} \qquad (4.16)$$

Now

$$\int_0^t 1_{\{X_s=0\}} \Phi^2(v(s,X_s))\,ds = 0, \qquad (4.17)$$

because the expectation of the non-negative left-hand side gives

$$\int_0^t \int_{\{0\}} \Phi^2 \left(v(s,y)\right) v(s,y) dy ds = 0.$$

By (4.17), (4.16) and taking also into account again Corollary (1.9) Chapter VI of [24] we obtain that

$$L_t^Y \left(0\right) = \frac{1}{2} L_t^X \left(0\right).$$

Going back to (4.12) we obtain

$$\begin{cases} X_t = X_0 + \int_0^t \Phi \left(v \left(s, X_s\right)\right) dB_s^1 + \frac{1}{2} L_t^X \left(0\right), \\ v \left(t, .\right) = \text{law density of } X_t,\ t \in [0,T], \\ v(0, .) = u_0. \end{cases} \tag{4.18}$$

Taking into account (4.14) we obtain

$$\begin{cases} X_t = X_0 + \int_0^t \Phi \left(v \left(s, X_s\right)\right) dB_s + \frac{1}{2} L_t^X \left(0\right), \\ v \left(t, .\right) = \text{law density of } X_t, \\ v(0, \cdot) = u_0. \end{cases} \tag{4.19}$$

This happens because $\int_0^t \Phi(v(s, X_s)) dB_s^2 = 0$. In fact its expectation gives $\mathbb{E}\left(\int_0^T \Phi^2(v(s, X_s)) ds\right) = 0$ since for each $s \in [0,T]$, X_s has a density. The proof is now concluded.

5 The probabilistic representation

A solution of (4.19) is the candidate for a probabilistic representation of a solution to (4.18). The proposition below gives an illustration of this, even though this constitutes the *easy* part of the probabilistic representation.

We suppose again the validity of Assumptions 1. and 2.

Proposition 13. *Let X be a solution in law of*

$$\begin{cases} X_t = X_0 + \int_0^t \chi_v \left(s, X_s\right) dB_s + \frac{1}{2} L_t^X \left(0\right), \\ v \left(t, .\right) = \textit{law density of } X_t,\ t \in [0,T], \\ \chi_v \left(s, x\right) \in \Phi \left(v \left(s, x\right)\right), \quad ds \otimes dx \ \textit{a.e.} \\ X \geq 0, \\ v(0, \cdot) = u_0. \end{cases} \tag{5.20}$$

Then v is the generalized solution in the sense of Definition 2 to equation (1.1).

Remark 14. *i) Equation (5.20) is formalized by*

$$\begin{cases} X_t \in X_0 + \int_0^t \Phi\left(v\left(s, X_s\right)\right) dB_s + \dfrac{1}{2} L_t^X\left(0\right), \\ v\left(t, .\right) = \textit{law density of } X_t,\ t \in [0,T], v(0,\cdot) = 0, \\ X \geq 0. \end{cases} \tag{5.21}$$

ii) If Φ is continuous (5.21). coincide with (4.19).

iii) According to Corollary 8 v is also the weak solution of (4.18) in the sense of Definition 4.

Proof. Let $\varphi \in C_0^\infty\left(\mathbb{R}_+\right)$ be such that $\varphi'\left(0\right) = 0$. By Itô formula we obtain

$$\begin{aligned} \varphi\left(X_t\right) &= \varphi\left(X_0\right) + \int_0^t \varphi'\left(X_s\right) \chi_v\left(s, X_s\right) dB_s \\ &\quad + \frac{1}{2} \int_0^t \varphi''\left(X_s\right) \chi_v^2\left(s, X_s\right) ds + \int_0^t \varphi'\left(X_s\right) dL_s^X\left(0\right). \end{aligned} \tag{5.22}$$

The last term above gives

$$\int_0^t \varphi'\left(X_s\right) dL_s^X\left(0\right) = \int_0^t 1_{\{X_s = 0\}} \varphi'\left(X_s\right) dL_s^X\left(0\right) + \int_0^t 1_{\{X_s \neq 0\}} \varphi'\left(X_s\right) dL_s^X\left(0\right).$$

The first integral of the right-hand side equals

$$\varphi'\left(0\right) \int_0^t 1_{\{X_s = 0\}} dL_s^X\left(0\right) = 0,$$

by the assumptions on φ.

The absolute value of the second integral is bounded by

$$\left\|\varphi'\right\|_\infty \int_0^t 1_{\{X_s \neq 0\}} dL_s^X\left(0\right) = 0,$$

by Theorem 7.1 from [20]. So we can conclude that $\int_0^t \varphi'\left(X_s\right) dL_s^X\left(0\right) = 0$.

Now, by taking the expectation in (5.22), for $t \in [0,T]$, we obtain

$$\begin{aligned} \int_0^\infty \varphi\left(x\right) v\left(t,x\right) dx &= \int_0^\infty \varphi\left(x\right) u_0\left(x\right) dx \\ &\quad + \frac{1}{2} \int_0^t \int_0^\infty \varphi''\left(x\right) v\left(s,x\right) \chi_v^2\left(s,x\right) dx ds. \end{aligned}$$

Since $\beta(u) = u\,\Phi^2(u)$, setting $\eta_v(s,x) = v(s,x)\,\chi_v^2(s,x)$, we have $\eta_v(s,x) \in \beta(v(s,x))$, $ds \otimes dx$ *a.e.* and v is a generalized solution. We remind that v is necessarily the unique solution of (1.1) and we can conclude the proof. □

Remark 15. *Any solution to equation*

$$\begin{cases} dX_t \in \Phi(v(t,X_t))\,dB_t + \frac{1}{2}dL_t^X(0), \\ v(t,.) = \textit{law density of } X_t,\ t \in [0,T], \\ v(0,.) = u_0, \end{cases}$$

solves also the reflecting Skorohod problem

$$\begin{cases} dX_t \in \Phi(v(t,X_t))\,dB_t + dK_t, \\ K \textit{ is increasing}, \\ X \geq 0, \int_0^T X_s dK_s = 0, \end{cases} \tag{5.23}$$

with $L_t^X(0) = 2K_t$.

Indeed, $(L_t^X(0))$ *is an increasing process, being a limit of increasing processes (see Corollary 1.9 Chapter VI of [24]) and* $\int_0^t 1_{\{X_s=0\}} dL_s^X(0) = 0$ *by Theorem 7.1 from [20].*

In particular showing the existence of a solution to (5.21)*, also establishes the existence for equation (5.23).*

The consideration of the first part of Section 3 allows to state effectively the existence of solutions for (5.20) which consists in a sort of converse statement of Proposition 13.

Theorem 16. *Under Assumptions 1 and 2 there is a solution in law of (5.20).*

Proof. The proof is the object of Section 4 with some obvious adaptations to the case when Φ was not continuous and it has to be associated with a graph.

□

Remark 17. *The conclusion of Theorem 16 is valid under other settings of similar hypotheses (but different) as those in Assumptions 1 and 2. For instance under the one of the following assumptions, there is a solution in law to* (5.20)*.*

- *Suppose* Φ *to be continuous on* $\mathbb{R}_+$ *and Assumption 1. ii) is valid. Then the conclusion of Theorem 16 still holds if in Assumption 1. we replace* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *with* $u_0 \in L^1(\mathbb{R})$*. In particular, there is a solution in law to to* (5.20)*. Indeed by Theorem 2.6 of [5], a probabilistic representation on the whole line takes place. This constitutes the replacement tool to Theorem 4.4 of [11] and of Theorem 5.4 of [4], mentioned in the lines before* (4.10)*.*

- *Item iii) in Assumption 1. is somehow technical and it can be often relaxed for instance when* β *is non-degenerate and* Φ *is smooth. Typically, when* Φ *is of of class* C^3*, Lipschitz,* β *is non-degenerate and* u_0 *is absolutely continuous with derivative in* $H^{2+\alpha}$ *for some* $0 < \alpha < 1$*. Then there is a (even a strong) solution of* (4.10)*, see for instance Proposition 2.2 in [19]. As for the previous item, by the same proof as for Theorem 16, there will be a solution to* (5.20)*.*

6 Appendix

Proof of Proposition 5.

Let $\varphi \in C_0^\infty(\mathbb{R}_+)$. Operating by integration by parts, (2.5) it is equivalent to

$$\begin{aligned} \int_0^\infty \varphi(x)\, u(t,x)\, dx &= \int_0^\infty \varphi(x)\, u_0(x)\, dx \\ &\quad - \int_0^t \varphi'(0)\, \eta_u(s,0)\, ds + \int_0^t \int_0^\infty \varphi''(x)\, \eta_u(s,x)\, dxds. \end{aligned} \tag{6.24}$$

We observe that (u, η_u) is a generalized solution if (6.24) holds for every $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi'(0) = 0$.

We can easily see that if (u, η_u) is a weak solution then it is obviously also a generalized solution.

In the other sense we assume that (u, η_u) be a generalized solution. To show that it is also a weak solution we need to show that (2.5) holds for every $\varphi \in C_0^\infty(\mathbb{R}_+)$. We know in fact that (6.24) holds for every $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi'(0) = 0$.

Let $\varepsilon > 0$. We consider a sequence $\chi_\varepsilon(x) : \mathbb{R}_+ \to [0,1]$ smooth such that

$$\chi_\varepsilon(x) = \begin{cases} 0 & : x \in [0, \varepsilon) \\ 1 & : x > 2\varepsilon. \end{cases}$$

We set

$$\varphi_\varepsilon(x) = \int_0^x \chi_\varepsilon(y)\, \varphi'(y)\, dy + \varphi(0) + c(\varepsilon)$$

where

$$c(\varepsilon) = \int_0^\infty (1 - \chi_\varepsilon)(y)\, \varphi'(y)\, dy.$$

We note that φ_ε is obviously smooth. Moreover it has compact support since for $x > \max(\text{supp}\varphi)$ we have

$$\varphi_\varepsilon(x) = \int_0^\infty \chi_\varepsilon(y)\, \varphi'(y)\, dy + \varphi(0) + c(\varepsilon) = 0.$$

By assumption, (6.24) holds with φ replaced by φ_ε. Obviously $\varphi_\varepsilon \to \varphi$ pointwise and $\varphi'_\varepsilon \to \varphi'$ in $L^p(\mathbb{R})$, for all $p \geq 1$.

Since $\varphi'_\varepsilon(0) = 0$, (2.5) holds for φ replaced by φ_ε, i.e.

$$\begin{aligned} \int_0^\infty \varphi_\varepsilon(x)\, u(t,x)\, dx &= \int_0^\infty \varphi_\varepsilon(x)\, u_0(x)\, dx \\ &\quad - \int_0^t \int_0^\infty \varphi'_\varepsilon(x)\, \eta'_u(s,x)\, dxds. \end{aligned} \tag{6.25}$$

We remark that $c(\varepsilon) \to 0$ for $\varepsilon \to 0$ and

$$|\varphi_\varepsilon(x)| \leq \int_0^\infty \left|\varphi'(y)\right| dy + \varphi(0) + \sup_\varepsilon c(\varepsilon).$$

Let $t \in [0,T]$. By the dominated convergence theorem the left-hand side of (6.25) (respectively the first integral in the right-hand side), converges to $\int_0^\infty \varphi(x)\, u(t,x)\, dx$ (respectively $\int_0^\infty \varphi(x)\, u_0(x)\, dx$).

We observe

$$\left|\varphi'_\varepsilon(x)\right| \leq \left|\varphi'(x)\right| \mathbf{1}_I(x),$$

I being a compact interval including the support of φ.

Again by dominated convergence theorem the second integral in the right-hand side of (6.25), converges to $\int_0^t \int_0^\infty \varphi'(x)\, \eta'_u(s,x)\, dxds$. This shows (2.5) for (1.1) and conclude the proof.

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References

[1] N. D. Alikakos and R. Rostamian. Large time behavior of solutions of Neumann boundary value problem for the porous medium equation. *Indiana Univ. Math. J.*, 30(5):749–785, 1981.

[2] V. Barbu. *Partial differential equations and boundary value problems*, volume 441 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1998. Translated and revised from the 1993 Romanian original by the author.

[3] V. Barbu. *Nonlinear differential equations of monotone types in Banach spaces*. Springer Monographs in Mathematics. Springer, New York, 2010.

[4] V. Barbu, M. Röckner, and F. Russo. Probabilistic representation for solutions of an irregular porous media type equation: the degenerate case. *Probab. Theory Related Fields*, 151(1-2):1–43, 2011.

[5] N. Belaribi, F. Cuvelier, and F. Russo. A probabilistic algorithm approximating solutions of a singular pde of porous media type. *Monte Carlo Methods Appl.*, 17(4):317–369, 2011.

[6] N. Belaribi, F. Cuvelier, and F. Russo. Probabilistic and deterministic algorithms for space multidimensional irregular porous media equation. *SPDEs: Analysis and Computations*, 1(1):3–62, 2013.

[7] N. Belaribi and F. Russo. About Fokker-Planck equation with measurable coefficients: application to the fast diffusion equation. *Elect. Journal Probab.*, 17(84):1–28, 2012.

[8] S. Benachour, P. Chassaing, B. Roynette, and P. Vallois. Processus associés à l'équation des milieux poreux. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(4):793–832 (1997), 1996.

[9] P. Benilan, H. Brezis, and M. G. Crandall. A semilinear equation in $L^1(\mathbb{R}^N)$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 2(4):523–555, 1975.

[10] P. Benilan and M. G. Crandall. The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$. *Indiana Univ. Math. J.*, 30(2):161–177, 1981.

[11] P. Blanchard, M. Röckner, and F. Russo. Probabilistic representation for solutions of an irregular porous media type equation. *Ann. Probab.*, 38(5):1870–1900, 2010.

[12] L. Bo and Y. Wang. On one-dimensional reflecting stochastic differential equations with non-Lipschitz coefficients. Preprint.

[13] H. Brezis and M. G. Crandall. Uniqueness of solutions of the initial-value problem for $u_t - \Delta\varphi(u) = 0$. *J. Math. Pures Appl. (9)*, 58(2):153–163, 1979.

[14] H. Brézis and W. A. Strauss. Semi-linear second-order elliptic equations in L^1. *J. Math. Soc. Japan*, 25:565–590, 1973.

[15] E. Cépa. Équations différentielles stochastiques multivoques. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(10):1075–1078, 1994.

[16] M. Chaleyat-Maurel and N. El Karoui. Un problème de réflexion et ses applications au temps local et aux équations différentielles stochastiques sur $\mathbb{R}$-cas continu. In *Temps locaux*, volume 52–53 of *Astérisque*, pages 117–144. 1978.

[17] G. da Prato and V. Barbu. The Neumann problem on unbounded domain of $\mathbb{R}^d$ and stochastic variational inequalities. *Preprint No. 14/ 2005. ZiF Research group.*

[18] C. Graham, Th. G. Kurtz, S. Méléard, Ph. E. Protter, M. Pulvirenti, and D. Talay. *Probabilistic models for nonlinear partial differential equations*, volume 1627 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996. Lectures given at the 1st Session and Summer School held in Montecatini Terme, May 22–30, 1995, Edited by Talay and L. Tubaro, Fondazione C.I.M.E. [C.I.M.E. Foundation].

[19] B. Jourdain and S. Méléard. Propagation of chaos and fluctuations for a moderate model with smooth initial data. *Ann. Inst. H. Poincaré Probab. Statist.*, 34(6):727–766, 1998.

[20] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[21] P.-L. Lions and A.-S. Sznitman. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.*, 37(4):511–537, 1984.

[22] H. P. Jr. McKean. Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)*, pages 41–57. Air Force Office Sci. Res., Arlington, Va., 1967.

[23] É. Pardoux and R. J. Williams. Symmetric reflected diffusions. *Ann. Inst. H. Poincaré Probab. Statist.*, 30(1):13–62, 1994.

[24] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[25] A. Rozkosz and L. Słomiński. Stochastic representation of reflecting diffusions corresponding to divergence form operators. *Studia Math.*, 139(2):141–174, 2000.

[26] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*, volume 49 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[27] D. W. Stroock and S. R. S. Varadhan. Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.*, 24:147–225, 1971.

[28] A. S. Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.