

Blending verification in domain decomposition methods to achieve high quality simulations without over-solving

Pierre Gosselet, Valentine Rey, Augustin Parret-Fréaud, Christian Rey

▶ **To cite this version:**

Pierre Gosselet, Valentine Rey, Augustin Parret-Fréaud, Christian Rey. Blending verification in domain decomposition methods to achieve high quality simulations without over-solving. 2nd ECCOMAS Young Investigators Conference (YIC 2013), Sep 2013, Bordeaux, France. hal-00855864

HAL Id: hal-00855864
https://hal.science/hal-00855864

Submitted on 30 Aug 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

YIC2013
Second ECCOMAS Young Investigators Conference
2–6 September 2013, Bordeaux, France

Blending verification in domain decomposition methods to achieve high quality simulations without over-solving

P. Gosselet [a,*], V. Rey [a], A. Parret-Fréaud [b], C. Rey [a]

[a] LMT-Cachan, ENS-Cachan/CNRS/UPMC/Pres UniverSud Paris
61 av du président Wilson, 94230 Cachan, France

[b] ENSTA ParisTech - Unité de Mécanique
828 boulevard des Maréchaux 91762 Palaiseau Cedex, France

*gosselet@lmt.ens-cachan.fr

Abstract. *This paper investigates the possibility to estimate the discretization error of large finite element problems solved by BDD or FETI domain decomposition methods. We show that a simple processing of the fields available during the iterations enables us to compute strict error bounds where we tell the difference between the effects of the FE discretization and the effects of the interface incompatibility. We then can stop iterations when the former prevail over the later and trigger mesh adaptation procedures.*

Keywords: Verification; domain decomposition methods; FETI; BDD; convergence criterion.

1 INTRODUCTION

Virtual testing is the industrialist's aim to minimize the number of experimental tests by replacing them by numerical simulations even for part of the certification process. To achieve that objective, we must propose computational methods for large systems which incorporate verification and adaptation tools.
A classical strategy for the parallel resolution of large systems is to employ solvers based on domain decomposition methods (DD), in particular we focus on FETI and BDD [3, 9, 10] methods which are now well mastered and remain scalable even for very complex problems [6, 15]. Schematically these methods consist in satisfying exactly the finite element equations inside subdomains whereas the continuity of the displacement and the balance of reactions on the interfaces are iteratively sought.
The a posteriori estimation of the error due to the finite element (FE) discretization can be done according several principles: equilibrium residual [1], flux projection [16] and error in constitutive relation [7]. A solution to obtain a strict estimation (an upper bound for the error), without involving hard to compute constants (related to the shape of the domain), is to make use of statically admissible fields (equilibrated residuals) which can be done by several techniques: in [5] a full dual analysis is conducted leading to an optimal estimator whereas in [11, 8, 13] the finite element stress is postprocessed which leads to lower quality estimations but at a much lower cost.
In [12] we proposed a way to process the fields obtained during FETI or BDD resolution [4] in order to enable a parallel reconstruction of (kinematic and static) admissible fields and a full parallel error estimation based on the error in constitutive equation. This estimation was possible even before convergence of the iterations associated to domain decomposition. We observed that the total error is due to the contribution of both the discretization inside subdomains and the discontinuities at the interface but that the latter quickly became negligible with respect to the former long before the domain decomposition method converged according to classical criterion.
Thanks to the results of [2] we are now able to propose a new strict error estimator for FETI and BDD domain decomposition methods which is the sum of a term purely associated to the interface error and an error dominated by the discretization error. This result enables us to propose an unbiased stopping criterion for domain decomposition.
The rest of this short paper is organized as follow: we first introduce notations and give the main results then we show the performance of the various estimators on a classical example.

2 NOTATIONS AND RESULTS

Let us consider the static equilibrium of a (polyhedral) structure which occupies the open domain $\Omega \subset \mathbb{R}^d$ and which is subjected to given body force $\underline{f}$ within Ω, to given traction force $\underline{g}$ on $\partial_g\Omega$ and to given displacement field $\underline{u}_d$ on the complementary part of the boundary $\partial_u\Omega \neq \varnothing$. We assume that the structure undergoes small perturbations and that the material is linear elastic, characterized by Hooke's elasticity tensor $\mathbb{H}$. Let $\underline{u}$ be the unknown displacement field, $\underline{\underline{\varepsilon}}(\underline{u})$ the symmetric part of the gradient of $\underline{u}$, $\underline{\underline{\sigma}}$ the Cauchy stress tensor. Let $\mathrm{KA} = \left\{\underline{u} \in \mathrm{H}^1(\Omega), \underline{u} = \underline{u}_d \text{on} \partial_u\Omega\right\}$ the set of admissible displacements and KA^0 the associated vector space. The problem to solve writes:

$$\begin{aligned}
&\text{Find } \underline{u}_{ex} \in \mathrm{KA},\ \underline{\underline{\sigma}} \in \left(\mathrm{L}^2(\omega)\right)^{d\times d}_{\text{sym}} \\
&\forall \underline{v} \in \mathrm{KA}^0(\Omega),\ \int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\Omega = \int_\Omega \underline{f}\cdot\underline{v} d\Omega + \int_{\partial_g\Omega} \underline{g}\cdot\underline{v} dS \\
&\underline{\underline{\sigma}} = \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_{ex})
\end{aligned} \tag{1}$$

Let $\underline{u}_H$ be a finite element approximation of previous problem, $\underline{\underline{\sigma}}_H = \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_H)$ be the associated stress and $\hat{\underline{\underline{\sigma}}}_H$ be a post processed field which satisfies all equations except the constitutive equation, we then have the following bound for the energy norm of the error [7]:

$$|||\underline{u}_{ex} - \underline{u}_H|||^2_\Omega \leqslant \mathrm{e}^2_{\mathrm{CR}_\Omega}(\underline{\mathrm{u}}_\mathrm{H}, \hat{\underline{\underline{\sigma}}}_\mathrm{H}) = \int_\Omega \left(\mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_H) - \hat{\underline{\underline{\sigma}}}_H\right) : \mathbb{H}^{-1} : \left(\mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_H) - \hat{\underline{\underline{\sigma}}}_H\right) \tag{2}$$

Let us now consider the resolution of previous finite element problem by a domain decomposition method. Let $(\Omega^{(s)})$ be a partition of Ω. The principle of domain decomposition method is to satisfy at any time the conservation of momentum independently on subdomains whereas the continuity of displacement $\underline{u}^{(s)} = \underline{u}^{(s)}$ and the action-reaction principle $\underline{\underline{\sigma}}^{(s)} \cdot n^{(s)} + \underline{\underline{\sigma}}^{(s')} \cdot n^{(s')} = 0$ on $\partial\Omega^{(s)} \cap \partial\Omega^{(s)}$ are alternatively satisfied. During any iteration of the DD solver we can deduce the following fields:

- $\underline{u}_D$: a displacement field continuous at the interface, hence kinematically admissible on the whole domain Ω, which solves local Dirichlet problems.
- $\underline{u}_N$: a displacement field so that the associated stress field $\underline{\underline{\sigma}}_N$ is equilibrated on the interface, which solves local Neumann problems and which can be the input to the parallel computation of a stress field satisfying the global equilibrium $\hat{\underline{\underline{\sigma}}}_N$.

We then have the following bounds [12, 14]:

$$\begin{aligned}
|||\underline{u}_{ex} - \underline{u}_D|||_\Omega &\leqslant \sqrt{\sum_s \mathrm{e}^2_{\mathrm{CR}_\Omega}(\underline{\mathrm{u}}^{(\mathrm{s})}_\mathrm{D}, \hat{\underline{\underline{\sigma}}}^{(\mathrm{s})}_\mathrm{N})} \\
|||\underline{u}_{ex} - \underline{u}_N|||_\Omega &\leqslant \sqrt{\sum_s \mathrm{e}^2_{\mathrm{CR}_\Omega}(\underline{\mathrm{u}}^{(\mathrm{s})}_\mathrm{N}, \hat{\underline{\underline{\sigma}}}^{(\mathrm{s})}_\mathrm{N})} + |||\underline{u}_D - \underline{u}_N|||_\Omega
\end{aligned} \tag{3}$$

The second term of the second expression can be expressed in term of a specific norm of the residual of the domain decomposition method which we note $\sqrt{\mathbf{r}^T\mathbf{z}}$.

3 ASSESSMENTS

Let us consider the structure of Figure 1 first proposed in [11]. We impose homogeneous Dirichlet boundary conditions in the central hole and on the base. The smaller hole is subjected to a constant unit pressure p_0. A unit traction force $\underline{g}$ is applied normally to the surface on the left upper part. The remaining boundaries are traction-free. A crack is also initiated from the smaller hole. The structure is homogeneous linear elastic (unit Young modulus) with Poisson coefficient of 0.3. We mesh the structure with first order triangular linear elements and create 16 subdomains by an automatic splitting of the mesh by Metis library, as shown in Figure 2. We use the FETI algorithm to solve the substructured problem with a Dirichlet preconditioner. The construction of statically admissible fields is performed using the Element Equilibration Technique (EET) with p+2-refinement.

Figure 3(a) presents the estimator of [12] and the new estimator during the DD iterations. They both converge close to the sequential estimator. On Figure 3(b) the two contributions to the new estimator are plotted, the DD-residual

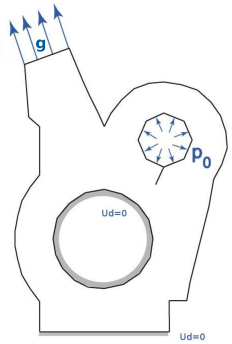
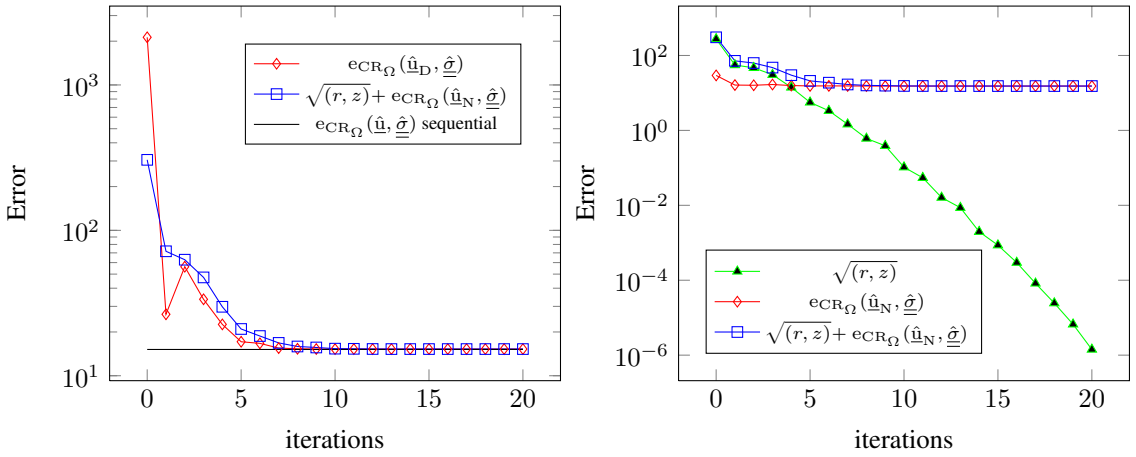


Figure 1: Cracked structure model problem



Figure 2: Decomposition in 16 subdomains



(a) Estimators for DD

(b) Separation of sources

Figure 3: Error estimates

decreases whereas the contribution bearing the discretization error is almost constant. We observe that the DD-iterations after the 6th do not improve the global solution while the DD-residual is only 100 times smaller than the initial residual (a classical stopping criterion would correspond to a ratio of 10^6). Moreover Figure 4 enables us to verify that the map of the element contribution to the error estimator is converged when the algebraic error is negligible (10 times smaller) with respect to the discretization error: the maps represent the difference between the final element contributions (iteration 20) and the current element contributions at iterations 1 (initialization) and 7 (algebraic error is negligible). This means that the map at iteration 7 is sufficient to be used as an input to a mesh adaptation procedure.

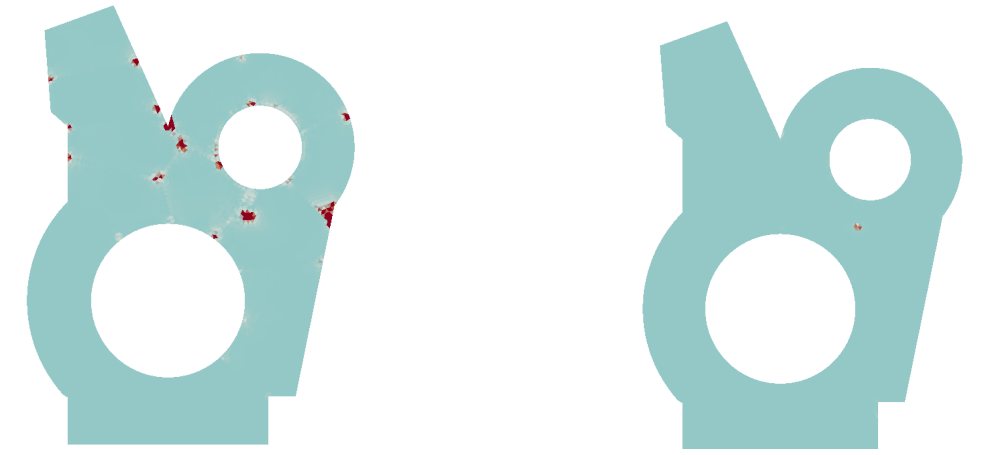
4 CONCLUSION

Verification can be easily conducted in parallel when using a FETI or BDD domain decomposition method. The quality of the parallel estimator is very close to the sequential one with CPU cost divided by the number of processors. Moreover, a new bound enables to separate the contributions of the discretization and of the DD residual. This enables us to give an unbiased stopping criterion for the DD when the global error is no more minimized by the iterations and mesh adaptation is required.

REFERENCES

[1] I. Babuška and W. C. Rheinboldt. Error estimates for adaptative finite element computation. *SIAM, Journal of Numerical Analysis*, 15(4):736–754, 1978.

[2] Alexandre Ern, Serge Nicaise, and Martin Vohralík. An accurate flux reconstruction for discontinuous Galerkin approximations of elliptic problems. *Comptes Rendus Mathematique*, 345(12):709712, 2007.



(a) $|e_{\mathrm{CR}_\Omega}(\hat{\underline{u}}_N, \hat{\underline{\sigma}})_{it=1} - e_{\mathrm{CR}_\Omega}(\hat{\underline{u}}_N, \hat{\underline{\sigma}})_{it=20}|$ (b) $|e_{\mathrm{CR}_\Omega}(\hat{\underline{u}}_N, \hat{\underline{\sigma}})_{it=7} - e_{\mathrm{CR}_\Omega}(\hat{\underline{u}}_N, \hat{\underline{\sigma}})_{it=20}|$

Figure 4: Convergence of the element contribution to the estimator

[3] C. Farhat and F. X. Roux. Implicit parallel processing in structural mechanics. *Computational Mechanics Advances*, 2(1):1–124, 1994. North-Holland.

[4] Pierre Gosselet and Christian Rey. Non-overlapping domain decomposition methods in structural mechanics. *Archives of Computational Methods in Engineering*, 13(4):515–572, 2006.

[5] Martin Kempeneers, Jean-François Debongnie, and Pierre Beckers. Pure equilibrium tetrahedral finite elements for global error estimation by dual analysis. *International Journal for Numerical Methods in Engineering*, 81(4):513–536, 2009.

[6] A. Klawonn, O. Rheinbach, and O. Widlund. An analysis of a feti-dp algorithm on irregular subdomains in the plane. *SIAM J. Numer. Anal.*, 46(5):2484–2504, 2008.

[7] P. Ladevèze. *Comparaison de modèles de milieux continus*. PhD thesis, Université P. et M. Curie, 1975.

[8] P. Ladevèze and D. Leguillon. Error estimate procedure in the finite element method and application. *SIAM Journal of Numerical Analysis*, 20(3):485–509, 1983.

[9] Patrick Le Tallec. Domain decomposition methods in computational mechanics. *Comput. Mech. Adv.*, 1(2):121–220, 1994.

[10] Jan Mandel. Balancing domain decomposition. *Communications in Numerical Methods in Engineering*, 9(3):233, 1993.

[11] N. Parés, P. Díez, and A. Huerta. Subdomain-based flux-free a posteriori error estimators. *Computer Methods in Applied Mechanics and Engineering*, 195(4-6):297–323, 2006.

[12] A. Parret-Fréaud, C. Rey, P. Gosselet, and F. Feyel. Fast estimation of discretization error for fe problems solved by domain decomposition. *Computer Methods in Applied Mechanics and Engineering*, 199(49-52):3315–3323, 2010.

[13] F. Pled, L. Chamoin, and P. Ladevèze. On the techniques for constructing admissible stress fields in model verification: Performances on engineering examples. *International Journal for Numerical Methods in Engineering*, 88(5):409–441, 2011.

[14] Valentine Rey, Pierre Gosselet, and Christian Rey. A guaranteed error bound separating algebraic and discretization contributions in non overlapping domain decomposition method. *submitted*, 2013.

[15] N. Spillane, V. Dolean, P. Hauret, F. Nataf, and D.J. Rixen. Solving generalized eigenvalue problems on the interfaces to build a robust two level feti method. *submitted, available on HAL*, 2012.

[16] O. C. Zienkiewicz and J. Z. Zhu. A simple error estimator and adaptive procedure for practical engineerng analysis. *International Journal for Numerical Methods in Engineering*, 24(2):337–357, 1987.