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Abstract

We solve a linear robust problem with mixed-integer first-stage variables and continuous second stage variables. We consider column wise uncertainty. We first focus on a problem with right hand-side uncertainty which satisfies a "full recourse property" and a specific definition of the uncertainty. We propose a solution based on a generation constraint algorithm. Then we give some generalizations of the approach: for left-hand side uncertainty and for uncertainty sets defined by a polytope. Finally we solve the problem when the "full recourse property" is not satisfied.

1 Introduction

This paper deals with robust mixed-integer linear programming (MILP) to study problems with uncertain data. This is a possible alternative to two-stage stochastic linear programming introduced by Dantzig in [8]. In this framework the uncertain data of the problem are modeled by random variables, and the decision-maker looks for an optimal solution with respect to the expected objective value. He makes decisions in two stages: first before discovering the actual value taken by the random variables, second once uncertainty has been revealed. However, this approach requires to know the underlying probability distribution of the data, which is, in many cases, not available; furthermore the size of the resulting optimization model increases in such a way that the stochastic optimization problem is often not tractable. Robust optimization is a recent approach that does not rely

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on a prerequisite precise probability model but on mild assumptions on the uncertainties involved in the problem, as bounds or reference values of the uncertain data. It looks for a solution that remains satisfactory for all realizations of the data (i.e. for worst scenarios). It was first explored by Soyster [12] who proposed a linear optimization model for data given in a convex set. However this is an over conservative approach that leads to optimal solutions too far from the one of the nominal problem. Robust adjustable optimization models have been proposed and studied to address this conservatism. More precisely, a lot of recent published works cover robust linear programming with row-wise uncertainty for continuous variables [3, 4, 6, 7] or discrete variables [1, 10] and, even more recently, column-wise right-hand side uncertainty [5, 11, 13].

In [9], Gabrel et al. propose a solution based on the approaches given in these last papers to solve a location transportation problem. We first show that their solution can be applied to any linear program with mixed-integer first stage variables and continuous recourse variables. We will also see that problems with left-hand side uncertainty can be solved in the same way. Then, we show that the method can also be used for an affine definition of the uncertainty set which is more general than the one used in [9, 11, 13].

To the extent of our knowledge, in all works published until now, the authors always assumed that the problem satisfies a "full recourse property" (see Section 2) which cannot be always satisfied for real problems: we show that, when this property is not verified, we can modify the objective function in order to use the previous approach to solve the problem. Some tests on a simple production problem prove the feasibility of our approach.

We focus here on a linear robust problem with right-hand and left hand-side uncertainty, mixed-integer first-stage variables and continuous second-stage variables. In Section 2 we present the general problem. For the sake of clarity, we first study the robust problem with right-hand side uncertainty and full recourse property with a specific definition of the uncertainty set. In Section 3 we show how to modelize and solve the recourse problem. In Section 4 we present the solution for the robust problem. In Section 5 we show that our results can be applied in case of left-hand side uncertainty and we extend our results to other definitions of the uncertainty set. Finally, in Section 6 we study the cases where the full recourse property is not verified. To facilitate reading the article, we present in Annex 1 and 2 two proofs of results given in Section 6.1 .

2 A mixed-integer linear robust problem

We consider applications requiring decision-making under uncertainty which can be modeled as a two-stage mixed-integer linear program with recourse. The set of variables is partitioned into two distinct sets: the x variables, called decision variables, concern the decisions to be taken in the first stage, before knowing the realization of the uncertain events; the second stage variables y, called recourse variables, will be fixed only after the uncertainty has been revealed.

We focus here on robust mixed-integer linear problems when the constraints coefficients are uncertain, as well on the right-hand side as on the left-hand side. In addition, we restrict our study to the case where the recourse variables y are continuous variables while the decision variables x are mixed-integer variables.

The deterministic problem can be formulated as the following MILP (in this paper we will omit the transpose sign ^{tr} when there is no possible confusion):

$$P \left| \begin{array}{ll} \min\limits_{x,y} \alpha x + \beta y & \\ \quad Ax + By \geq d & (1) \\ \quad Cx \geq b & (2) \\ \quad x_i \in \mathbb{N},\ i = 1, ..., p_1,\ x_i \in \mathbb{R}_+,\ i = (p_1 + 1), ..., p,\ y \in \mathbb{R}^q_+. & (3) \end{array} \right.$$

where $A \in \mathbb{Q}^{T \times p},\ B \in \mathbb{Q}^{T \times q},\ d \in \mathbb{Q}^T, C \in \mathbb{Q}^{n \times p},\ b \in \mathbb{Q}^n, \alpha \in \mathbb{Q}^p_+,\ \beta \in \mathbb{Q}^q_+$, and $\mathbb{Q}$ is the set of rational numbers.

We assume that there exists (x, y) such that (1)-(3) are satisfied and we say that a solution x is feasible if x satisfies constraints (2) and (3). The uncertain coefficients are those of d (right-hand side), A and B (left-hand side).

We assume that the program P satisfies the property $\mathcal{P}$, called "full recourse property": for any feasible values of the decision variables (here x) and for any possible value of A,B and d, there exist values of the recourse variables (here y) such that (1) is satisfied, that is such that there exists a feasible solution of P. Let us notice that the property $\mathcal{P}$ is always satisfied if there is a column of B whose all terms are positive. The hypothesis that P satisfies the property $\mathcal{P}$ cannot be always satisfied for real problems: we show in Section 6 that we can extend our results when $\mathcal{P}$ is not satisfied.

We suppose that d belongs to a given set $\mathfrak{D}$ which defines the set of possible scenarios and for the sake of clarity we assume at first that the uncertainty concerns only the right-hand side d of (1). We show in Section 5.1 that our results can be extended when the matrices A and B are also uncertain.

Our robustness objective is to find a feasible solution x, y of P that minimizes the total cost involved by the worst possible scenario of d in connection with x. We can state the robust problem as the following mathematical program:

$$PR \left| \begin{array}{ll} \min\limits_{x} & \alpha x \; + \; \max\limits_{d \in \mathfrak{D}} \min\limits_{y} \beta y \\ & \qquad\qquad\qquad By \geq d - Ax \\ & \qquad\qquad\qquad y \in \mathbb{R}^q_+ \\ & Cx \geq b \\ & x_i \in \mathbb{N},\ i = 1, ..., p_1,\ x_i \in \mathbb{R}_+,\ i = (p_1 + 1), ..., p. \end{array} \right.$$

For any feasible x, we define the following linear program $R(x)$ called "Recourse Program":

$$R(x) \left| \begin{array}{ll} \max\limits_{d \in \mathfrak{D}} \min\limits_{y} & \beta y \\ & By \geq d - Ax \\ & y \in \mathbb{R}^q_+. \end{array} \right.$$

Given a mathematical program π, we denote by $v(\pi)$ the value of an optimal solution. The robust program can then be rewritten as:

$$PR \left| \begin{array}{ll} \min\limits_{x} & \alpha x + v(R(x)) \\ & Cx \geq b \\ & x_i \in \mathbb{N},\ i = 1, ..., p_1,\ x_i \in \mathbb{R}_+,\ i = (p_1 + 1), ..., p. \end{array} \right.$$

Let us define more precisely the uncertainty. Following the idea proposed by Bertsimas and Sim [7] and Minoux [11], we suppose that each coefficient $d_t,\ t = 1, ..., T$ belongs to an interval $[\bar{d}_t - \Delta_t, \bar{d}_t + \Delta_t]$ where $\bar{d}_t$ is a given value and where $\Delta_t \geq 0$ is a given bound of the uncertainty of d. The uncertainty set $\mathfrak{D}$ is therefore given by:

$$\mathfrak{D} = \{d :\ d_t \in [\bar{d}_t - \Delta_t, \bar{d}_t + \Delta_t],\ \forall t = 1, ..., T\}.$$

For a fixed x, the worst scenario is obtained for $d_t = \bar{d}_t + \Delta_t,\ t = 1, ..., T$. Indeed, $By \geq \bar{d} + \Delta - Ax$ implies $By \geq d - Ax$ for all $d \in \mathfrak{D}$. Thus this uncertainty definition brings the robust problem back to a deterministic one. It provides a high "protection" against uncertainty, but it is very conservative in practice and leads to very expensive solutions. To avoid overprotecting the system, we impose, as in [13], the constraint

$$\sum_{\substack{t=1 \\ \Delta_t > 0}}^{T} \left|\frac{d_t - \bar{d}_t}{\Delta_t}\right| \leq \bar{\delta},$$

where $\bar{\delta}$ is a positive integer which bounds the total scaled deviation of d from its nominal value $\bar{d}$. Notice that there always exists a worst scenario with $d_t \geq \bar{d}_t,\ \forall t$, hence we can redefine the uncertainty set $\mathfrak{D}$ as

$$\mathfrak{D} = \{d:\ d_t = \bar{d}_t + \delta_t \Delta_t,\ 0 \leq \delta_t \leq 1,\ \forall t = 1, ..., T,\ \sum_{t=1}^{T} \delta_t \leq \bar{\delta}\}.$$

3 The recourse problem

To solve the recourse "$max\ min$" problem for given values of the decision variables, the minimization linear sub-program is transformed in a maximization program by considering its dual. But that leads to a quadratic objective function. We show that whatever the coefficients in the recourse problem, the quadratic terms can be written as products of a 0-1 variable and a continuous but bounded variable, which allows a linearization of these products.

Let x be a feasible solution and let $d \in \mathfrak{D}$, we define the following linear program

$$R(x,d) \left| \begin{array}{l} \min\limits_{y}\ \beta y \\ \qquad By \geq d - Ax \\ \qquad y \in \mathbb{R}^q_+. \end{array} \right.$$

We notice that $R(x,d)$ has a finite solution for all feasible x and for all possible scenario d since P satisfies $\mathcal{P}$, and since $\beta y \geq 0$ for any feasible solution y of $R(x,d)$. Thus by the strong duality theorem, we have

$$v(R(x,d)) = v(DR(x,d)),$$

where $DR(x,d)$ is the dual program of $R(x,d)$:

$$DR(x,d) \left| \begin{array}{ll} \max\limits_{\lambda}\ (d - Ax)\lambda & \\ \qquad \lambda B \leq \beta & (4) \\ \qquad \lambda \in \mathbb{R}^T_+. & (5) \end{array} \right.$$

Then, for any feasible x, $v(R(x)) = \max_{d \in \mathfrak{D}} v(DR(x,d))$, and we can reformulate $DR(x)$ as

$$\max_{\substack{\delta:\ \sum_{t=1}^{T} \delta_t \leq \bar{\delta} \\ 0 \leq \delta_t \leq 1,\ t=1,...,T}} \ \max_{\substack{\lambda:\ \lambda B \leq \beta \\ \lambda \in \mathbb{R}^T_+.}} \sum_{t=1}^{T} [(\bar{d}_t + \delta_t \Delta_t - (Ax)_t)\lambda_t]$$

where for a vector (u), we denote by $(u)_t$ the t-th coordinate of (u).

$DR(x)$ can be written as follows:

$$DR(x) \left| \begin{array}{ll} \max\limits_{\lambda,\delta} \sum\limits_{t=1}^{T} [(\bar{d}_t - (Ax)_t)\lambda_t + \Delta_t \delta_t \lambda_t] & \\ \quad \lambda B \leq \beta & (4) \\ \quad \sum\limits_{t=1}^{T} \delta_t \leq \bar{\delta} & (6) \\ \quad 0 \leq \delta_t \leq 1 \; t = 1, ..., T & (7) \\ \quad \lambda \in \mathbb{R}_+^T. & (5) \end{array} \right.$$

However, this bilinear program with linear constraints is not concave. Therefore computing the optimal solution of $DR(x)$ written as above is not an easy task. We now prove that we can solve $DR(x)$ by solving an equivalent mixed-integer linear program. To prove this claim, we need the following proposition:

Proposition 1. *There is an optimal solution λ^*, δ^* of $DR(x)$ such that $\delta_t^* \in \{0, 1\},\ 1 \leq t \leq T$.*

Proof. For any fixed λ, there is an optimal solution, (λ, δ^*), of $DR(x)$, where δ^* is an extreme point of the polyhedron defined by (6) and (7), that is to say, a point such that $\delta_t^* \in \{0, 1\},\ 1 \leq t \leq T$, since $\bar{\delta}$ is an integer.
More precisely $\delta_t^* = 1$ for indices corresponding to the $\bar{\delta}$ largest $\Delta_t \lambda_t$. □

Therefore we can assume that there is an optimal solution of $DR(x)$, such that $\lambda_t \delta_t$ belongs to $\{0, \lambda_t\}$. To linearize $\lambda_t \delta_t$, we now prove that we can restrict ourselves to the case where λ_t is bounded by a constant Λ, for all t.

Proposition 2. *There exists $\Lambda > 0$ which can be calculated from B and β such that the conditions $\lambda_t \leq \Lambda,\ t = 1, ..., T$, can be added to $DR(x)$ without loss of generality.*

Proof. Let x be feasible. Let us rewrite $DR(x)$ with the slack variables $\lambda'_t \geq 0,\ t = 1, ..., T$. The constraints (4) become: $B^{tr}\lambda + \lambda' = \beta$. Let $(\lambda^*, \lambda'^*, \delta^*)$ be an optimal solution of $DR(x)$, we can assume w.l.o.g. that (λ^*, λ'^*) is an optimal basic solution of $DR(x)$ when δ is set to δ^*.
Therefore, there exists a basic matrix $E = (e_{ij})$ of $(B^{tr}\ I_T)$ and basic vectors $\lambda_E^*, \lambda_E'^*$ such that: $(\lambda_E^*\ \lambda_E'^*)^{tr} = E^{-1}\beta$. Let $\hat{e}$ be an upper bound on the absolute value of the coefficients of E^{-1} for all basic matrices E of $(B^{tr}\ I_T)$, and let $\hat{\beta} = \max\limits_{i=1,...,q} \beta_i$, we have $\lambda_t^* \leq \hat{e}\hat{\beta}q,\ t = 1, ..., T$. Therefore there exists an optimal solution (λ^*, δ^*) of $DR(x)$ such that λ_t^* is bounded by $\Lambda = \hat{e}\hat{\beta}q$ for any $t = 1, ..., T$. □

We can now linearize $DR(x)$ by substituting the new variables ν_t to the products $\lambda_t\delta_t$ and by adding the constraints: $\nu_t \leq \lambda_t$, $\nu_t \leq \Lambda\delta_t$, $\nu_t \geq \lambda_t - \Lambda(1-\delta_t)$, $\nu_t \geq 0$.

$DR(x)$ is equivalent to the following mixed-integer linear program:

$$
LDR(x)\left|\begin{array}{ll}
\max\limits_{\lambda,\delta,\nu} & \sum_{t=1}^{T}[(\bar{d}_t - (Ax)_t)\lambda_t + \Delta_t\nu_t] \\
 & \lambda B \leq \beta \\
 & \sum_{t=1}^{T}\delta_t \leq \bar{\delta} \\
 & \nu_t \leq \lambda_t,\ t = 1,...,T \\
 & \nu_t \leq \Lambda\delta_t,\ t = 1,...,T \\
 & \lambda,\nu \in \mathbb{R}_+^T \\
 & \delta_t \in \{0,1\},\ t = 1,...,T.
\end{array}\right.
$$

Notice that the linearization constraints, $\nu_t \geq \lambda_t - \Lambda(1-\delta_t),\ t = 1,...,T$, can be omitted since the coefficients of ν_t in the objective function to maximize are positive.

4 Solving the robust problem

In order to solve the robust problem PR, we will first reformulate it as a linear program and then use a constraint generation algorithm. In the previous section, we proved that the recourse problem is equivalent to the linear program $LDR(x)$. Thus the robust problem can be reformulated as:

$$
PR\left|\begin{array}{ll}
\min\limits_{x} & \alpha x + v(LDR(x)) \\
 & Cx \geq b \\
 & x_i \in \mathbb{N},\ i = 1,...,p_1,\ x_i \in \mathbb{R}_+,\ i = (p_1+1),...,p.
\end{array}\right.
$$

Let $\mathcal{P}_Q$ be the polyhedron defined by the constraints of $LDR(x)$ where we replace $\delta_t \in \{0,1\}$ by $0 \leq \delta_t \leq 1$, and let $(\mathcal{P}_Q)_I = conv(\mathcal{P}_Q \cap \{\delta \in \mathbb{N}^m\})$, be the convex hull of the feasible solution of $LDR(x)$. Notice that this convex hull does not depend on x. $(\mathcal{P}_Q)_I$ is a polyhedron, thus we have

$$
LDR(x)\left|\begin{array}{ll}
\max\limits_{\lambda,\delta,\nu} & \sum_{t=1}^{T}[(\bar{d}_t - (Ax)_t)\lambda_t + \Delta_t\nu_t] \\
 & \begin{pmatrix}\lambda \\ \delta \\ \nu\end{pmatrix} \in (\mathcal{P}_Q)_I,
\end{array}\right.
$$

Let $\mathcal{S} = \{(\lambda^s, \delta^s, \nu^s)_{1\leq s\leq S}\}$, be the set of extreme points of $(\mathcal{P}_Q)_I$. For any feasible x, there is $s \in \{1,...,S\}$ such that $(\lambda^s, \delta^s, \nu^s)$ is an optimal solution of

Algorithm 1 Constraint generation algorithm

1: $(\lambda^0, \delta^0, \nu^0) = (0, 0, 0)$. Set $L \leftarrow -\infty, U \leftarrow +\infty, k \leftarrow 1$.

2: Solve the master problem :

$$PR^k \left| \begin{array}{ll} \min\limits_{x,z} & \alpha x + z \\ & z \geq \sum_{t=1}^{T} (\bar{d}_t - (Ax)_t)\lambda_t^s + \Delta_t \nu_t^s, \ 0 \leq s \leq k-1 \\ & Cx \geq b \\ & x_i \in \mathbb{N}, \ i = 1, ..., p_1, \ x_i \in \mathbb{R}_+, \ i = (p_1+1), ..., p \\ & z \in \mathbb{R} \end{array} \right.$$

Let (x^k, z^k) be the obtained solution.
$L \leftarrow \alpha x^k + z^k$.

3: Solve $LDR(x^k)$. Let $(\lambda^k, \delta^k, \nu^k)$ be the optimal solution.

$$U \leftarrow \min\{U, \alpha x^k + v(DR(x^k))\}.$$

if $U = L$, **then** return (x^k, z^k) **else** go to 4.

4: Add the constraint

$$z \geq \sum_{t=1}^{T} (\bar{d}_t - (Ax)_t)\lambda_t^k + \Delta_t \nu_t^k,$$

to the master problem PR^k, $k \leftarrow k+1$ and go to 2.

$LDR(x)$.
Thus the robust problem can be reformulated as the linear program:

$$PR \left| \begin{array}{l} \min\limits_{x,z} \ \alpha x + z \\ \quad z \geq \sum_{t=1}^{T} [(\bar{d}_t - (Ax)_t)\lambda_t^s + \Delta_t \nu_t^s], \ 1 \leq s \leq S \\ \quad Cx \geq b \\ \quad x_i \in \mathbb{N}, \ i = 1, ..., p_1, \ x_i \in \mathbb{R}_+, \ i = (p_1+1), ..., p, \ z \in \mathbb{R} \end{array} \right. \tag{8}$$

However, due to the potentially tremendous number of constraints, we solve PR by a constraint generation algorithm as in [13] or [9]. Initially, we consider a subset $\mathcal{S}_0$ of $\mathcal{S}$; at a step k, we consider a subset $\mathcal{S}^k$ of $\mathcal{S}$ and we solve a relaxed program PR^k of PR, called *master problem*, which consists in solving PR with the subset of constraints (8) corresponding to $\mathcal{S}^k$. The obtained solution in denoted by (x^k, z^k).
Then we solve $DR(x^k)$, called *slave problem*, to check if (x^k, z^k) is optimal. If not, then a new constraint is added, i.e. an extreme point is added to $\mathcal{S}^k$ (See Algorithm 1).
On the basis that the number of extreme points of $(\mathcal{P}_Q)_I$ is finite, one can prove that this algorithm converges in a finite number of steps.

5 Some generalizations

5.1 Left-hand side uncertainty

In the previous sections, we assumed that the uncertainty concerned only the right-hand side d of constraints (1). We now prove that our approach can be generalized to the case where the constraint coefficients ($A = (A_{ti})_{1 \leq t \leq T,\ 1 \leq i \leq p}$), are also likely to be uncertain. As before, we assume that each coefficient A_{ti} belongs to an interval $[\bar{A}_{ti} - \Gamma_{ti}, \bar{A}_{ti} + \Gamma_{ti}]$, where $\bar{A}_{ti}$ is a given value and where Γ_{ti} is a given bound of the uncertainty of A_{ti}.
Furthermore, in order to avoid overprotecting the system, we assume that the total scaled deviation of the uncertainty of the i-th column of A, $A_i = (A_{ti},\ t = 1, ..., T)$, is bounded. Similarly to $\mathfrak{D}$, the uncertainty set $\mathcal{A}_i$ of A_i is defined as :

$$\mathcal{A}_i = \{A_i :\ A_{ti} = \bar{A}_{ti} - \gamma_{ti}\Gamma_{ti},\ 0 \leq \gamma_{ti} \leq 1, t = 1, ..., T,\ \sum_{t=1}^{T} \gamma_{ti} \leq \bar{\gamma}_i\},$$

where $\bar{\gamma}_i$ is a given integer.

The robust problem can thus be formulated as:

$$PR' \left| \begin{array}{lll} \min\limits_{x}\ \alpha x + \max\limits_{\substack{A_i \in \mathcal{A}_i, \\ \forall i = 1, ..., p \\ d \in \mathfrak{D}}}\ \min\limits_{y} & \beta y & \\ & By \geq d - (A_1, ..., A_p)x & \\ & y \in \mathbb{R}^q_+ & \\ Cx \geq b & & \\ x_i \in \mathbb{N},\ i = 1, ..., p_1, x_i \in \mathbb{R}_+,\ i = (p_1 + 1), ..., p. & & \end{array} \right.$$

And the recourse problem becomes:

$$DR'(x) \left| \begin{array}{l} \max\limits_{\lambda, \delta, \gamma}\ \sum_{t=1}^{T}[(\bar{d}_t - \sum_{i=1}^{p} \bar{A}_{ti} x_i)\lambda_t + \Delta_t \delta_t \lambda_t + \sum_{i=1}^{p} \Gamma_{ti} x_i \gamma_{ti} \lambda_t] \\ \quad \lambda B \leq \beta \\ \quad \sum_{t=1}^{T} \delta_t \leq \bar{\delta} \\ \quad 0 \leq \delta_t \leq 1,\ t = 1, .., T \\ \quad \lambda \in \mathbb{R}^T_+ \\ \quad \sum_{t=1}^{T} \gamma_{ti} \leq \bar{\gamma}_i,\ i = 1, ..., p \\ \quad 0 \leq \gamma_{ti} \leq 1,\ i = 1, .., p,\ t = 1, ..., T. \end{array} \right.$$

We can then linearize the quadratic terms ($\delta_t \lambda_t$ and $\gamma_{ti} \lambda_t$), to obtain a mixed-integer linear recourse problem and then solve the robust problem as we did in the previous sections.

Let us now consider that matrix B has uncertain coefficients. As before we assume that, for $t = 1, ..., T$ and $i = 1, ..., q$, $B_{ti} \in [\bar{B}_{ti} - \Phi_{ti}, \bar{B}_{ti} + \Phi_{ti}]$ and that column i of B is in the set $\{B_i :\ B_{ti} = \bar{b}_{ti} - \phi_{ti}\Phi_{ti},\ 0 \leq \phi_{ti} \leq 1,\ t = 1, ..., T,\ \sum_{t=1}^{T} \phi_{ti} \leq \bar{\phi}_i\}$.

The dual of the recourse problem is similar to the program $DR(x)$ defined in Section 3: we add to $DR(x)$ variables ϕ and constraints $\sum_{t=1}^{T} \phi_{ti} \leq \bar{\phi}_i,\ i = 1, ..., q$ and $0 \leq \phi_{ti} \leq 1,\ i = 1, ..., q,\ t = 1, ..., T$, and we replace constraints (4) by:

$$\sum_{t=1}^{T} (\bar{B}_{ti} - \phi_{ti}\Phi_{ti})\lambda_t \leq \beta_i,\ i = 1, ..., q \qquad (4').$$

Now there are quadratic terms $(\phi_{ti}\lambda_t)$ in the constraints. To linearized these terms, we must verify that the propositions 1 and 2 can be extended, i.e. there is an optimal solution such that $\phi \in \{0,1\}^{Tq}$ and λ is bounded. The proofs given in Section 3 can easily be extended. Let λ be fixed. If constraints $(4')$ and $\sum_{t=1}^{T} \phi_{ti} \leq \bar{\phi}_i,\ i = 1, ..., q$ are verified, a feasible integer solution can be obtained by setting ϕ_{ti} to 1 if i corresponds to one of the $\bar{\phi}_i$ larger values of $\Phi_{ti}\lambda_t$ and to 0 otherwise.

For Proposition 2, by using the same arguments as in Section 3, for any fixed ϕ we can bound λ_t by $\Lambda(\phi)$. Since $\phi_{ti} \in \{0,1\}$ and $\sum_{t=1}^{T} \phi_{ti} \leq \bar{\phi}_i$, there are 2^{qT} different values of ϕ and we can bound λ_t by $\max_\phi \Lambda(\phi)$.

5.2 Generalization to other uncertainty sets

In the previous sections, we assumed that uncertain coefficients could be written as $d_t = \bar{d}_t + \delta_t\Delta_t\ \forall t$, where δ_t expresses the uncertainty on d_t and satisfies $\sum_{t=1}^{T} \delta_t \leq \bar{\delta}$. Now we generalize our results when the vector d can be written as $d = \bar{d} + D\delta$, where the vector $\bar{d}$ and the matrix D are given and where δ belongs to a bounded polyhedron $\mathcal{D}$ whose extreme points $(d^1, ..., d^S)$ are known. Notice that this definition of the uncertainty covers the one given by Babonneau et al. in [2] Let us rewrite the recourse problem:

$$DR'(x) \left| \begin{array}{l} \max\limits_{\lambda,\delta}\ (\bar{d} + D\delta - Ax)\lambda \\ \qquad \lambda B \leq \beta \\ \qquad \delta \in \mathcal{D} \\ \qquad \lambda \in \mathbb{R}^T_+. \end{array} \right.$$

Let $v^1, ..., v^S \in [0,1]$ be variables such that $\delta = \sum_{s=1}^{S} d^s v^s$ and $\sum_{s=1}^{S} v^s = 1$. We can rewrite the recourse problem as

$$
DR'(x) \left|
\begin{array}{l}
\max\limits_{\lambda, v} \, (\bar{d} - Ax)\lambda + \sum\limits_{s=1}^{S} (v^s(Dd^s)\lambda) \\
\quad \lambda B \leq \beta \\
\quad \sum\limits_{s=1}^{S} v^s = 1 \\
\quad 0 \leq v^s \leq 1, \; s = 1, ..., S \\
\quad \lambda \in \mathbb{R}_+^T.
\end{array}
\right.
$$

Using the same argument as in Proposition 1, we can prove that there exists an optimal solution (λ^*, v^*) of $DR'(x)$ such that either $v^{s*} = 1$ or $v^{s*} = 0, \; s = 1, ..., S$. Therefore we can linearize the quadratic terms $v^s \lambda_t$, for all s and for all t, as we did in Section 3, to obtain a mixed-integer linear recourse problem, and finally we can solve the robust problem by using Algorithm 1.

6 The problem without the full recourse property

6.1 Solving the problem

In the previous sections, we assumed that the deterministic problem satisfied the property $\mathcal{P}$. We now prove that we can extend our results to the case where we only assume that the robust problem PR has a solution, i.e. there is x such that for all d in $\mathfrak{D}$ there is a feasible solution y to $R(x, d)$ and so, there is M such that $v(PR) \leq M$. In addition, the method detects if the problem has no solutions. For the sake of clarity, we give here the proof for the case where all the decision variables are integer. The complete proof for mixed integer decision variables is given in Annex 1. In Annex 2, we propose a general theoretical value of M; nevertheless, a better specific value of M can be calculated for each problem, as it is the case for the application presented in Section 6.2.

First we show how to obtain a new MILP, denoted P_ε such that the robust associated problem has the same optimal solution as the initial robust problem if there is one, and P_ε satisfies $\mathcal{P}$. To obtain P_ε, we add new recourse variables $w_t, \; t = 1, ..., T$. As in Sections 2, 3 and 4, for the sake of clarity and w.l.o.g., we consider only right-hand side uncertainty.

Let ε be a given strictly positive value, we define the following MILP:

$$
P_\varepsilon \left|
\begin{array}{ll}
\min\limits_{x,y,w} \alpha x + \beta y + \dfrac{M}{\varepsilon}\sum\limits_{t=1}^{T} w_t & \\
\quad Ax + By + w \geq d & (1\varepsilon) \\
\quad Cx \geq b & (2) \\
\quad x_i \in \mathbb{N},\ i = 1, ..., p & (3) \\
\quad y \in \mathbb{R}^q_+,\ w \in \mathbb{R}^T_+ & (4\varepsilon)
\end{array}
\right.
$$

Since the variables $w_t,\ t = 1, ..., T$, are not bounded, P_ε satisfies property $\mathcal{P}$.

We denote by PR_ε, the robust problem associated to P_ε,

$$
PR_\varepsilon \left|
\begin{array}{l}
\min\limits_{x}\ \alpha x\ +\ \max\limits_{d\in\mathfrak{D}} \min\limits_{y,w} \beta y + \frac{M}{\varepsilon}\sum_{t=1}^{T} w_t \\
\qquad\qquad\qquad\qquad By + w \geq d - Ax \\
\qquad\qquad\qquad\qquad y \in \mathbb{R}^q_+,\ w \in \mathbb{R}^T_+ \\
\qquad Cx \geq b \\
\qquad x_i \in \mathbb{N},\ i = 1, ..., p,
\end{array}
\right.
$$

and by $R_\varepsilon(x)$, $R_\varepsilon(x, d)$ and $DR_\varepsilon(x)$ the associated subproblems as those defined in Section 3. Notice that since all the inputs, $A, B, C, b, d, \alpha, \beta, \Delta$, of PR have rational coefficients, we can reduce PR_ε and all the corresponding subproblems to programs where all the inputs are integer. Therefore we assume from now that all the inputs are integer.

Proposition 3. *$v(PR_\varepsilon)$ satisfies* $0 \leq v(PR_\varepsilon) \leq v(PR) \leq M$.

Proof. Let $(\hat{x}, \hat{y})$ be an optimal solution of PR, By hypothesis, $v(PR) \leq M$ thus $v(R(\hat{x})) \leq M$. Let $\bar{d}$ be a scenario in $\mathfrak{D}$. Since $v(R(\hat{x})) = \max\limits_{d\in\mathfrak{D}} v(R(\hat{x}, d))$, we have $v(R(\hat{x}, \bar{d})) \leq M$. Let $\bar{y}$ be an optimal solution of $R(\hat{x}, \bar{d})$, we notice that $(y, w) = (\bar{y}, 0)$ is a feasible solution of $(R_\varepsilon(\hat{x}, \bar{d}))$ with the same cost. Thus $v(R_\varepsilon(\hat{x}, \bar{d})) \leq v(R(\hat{x}, \bar{d}))$, for any $\bar{d} \in \mathfrak{D}$, which implies $v(R_\varepsilon(\hat{x})) \leq v(R(\hat{x}))$, and $0 \leq v(PR_\varepsilon) \leq v(PR) \leq M$. □

Let (x^*, d^*, y^*, w^*) be an optimal solution of PR_ε: d^* is the worst scenario for x^*. Notice that Proposition 1 is valid for $DR_\varepsilon(x^*)$. Therefore $d^*_t = \bar{d}_t$ or $d^*_t = \bar{d}_t + \Delta_t$, and d^*_t is an integer. From Proposition 3, we have

$$
\alpha x^* + \beta y^* + \frac{M}{\varepsilon}\sum_{t=1}^{T} w^*_t \leq M.
$$

Since $\alpha x^* + \beta y^* \geq 0$, we have $\sum_{t=1}^{T} w_t^* \leq \varepsilon$, and thus $w_t^* \leq \varepsilon,\ \forall t = 1, .., T$.
We now prove that if (y^*, w^*) is a basic optimal solution of $(R_\varepsilon(x^*, d^*))$, then for ε small enough, we have $w_t^* = 0,\ \forall t$; and then (x^*, y^*) is feasible for PR, and therefore from Proposition 3, $v(PR) = v(PR_\varepsilon)$.

Let us rewrite PR_ε with the positive slack variables $\sigma = (\sigma_t,\ t = 1, ..., T)$: the constraint (1ε) becomes $Ax + By + w - \sigma = d$. Let $(x^*, d^*, y^*, w^*, \sigma^*)$ be an optimal solution where (y^*, w^*, σ^*) is a basic optimal solution of the program

$$R_\varepsilon(x^*, d^*) \left| \begin{array}{l} \min\limits_{y,w,\sigma} \beta y + \dfrac{M}{\varepsilon} \displaystyle\sum_{t=1}^{T} w_t \\ \qquad By + w - \sigma = d^* - Ax^* \\ \qquad y \in \mathbb{R}_+^q,\ \sigma, w \in \mathbb{R}_+^T. \end{array} \right.$$

which is equivalent to

$$R_\varepsilon(x^*, d^*) \left| \begin{array}{l} \min\limits_{y,w,\sigma} \beta y + \dfrac{M}{\varepsilon} \displaystyle\sum_{t=1}^{T} w_t \\ \qquad \begin{pmatrix} B & I_T & -I_T \end{pmatrix} \begin{pmatrix} y \\ w \\ \sigma \end{pmatrix} = d^* - Ax^* \\ \qquad y \in \mathbb{R}_+^q,\ \sigma, w \in \mathbb{R}_+^T. \end{array} \right.$$

Let $L = \begin{pmatrix} B & I_T & -I_T \end{pmatrix} = (l_{ij}) \in \mathbb{Z}^{T \times (q+2T)}$ and $l_M = \max_{i,j} |l_{ij}|$. We notice that L has rank T.

Proposition 4. *Let $\varepsilon < \frac{1}{(l_M)^T T^{T/2}}$. If (x^*, d^*, y^*, w^*) is an optimal solution of PR_ε then $w^* = 0$ and (x^*, d^*, y^*) is an optimal solution of PR.*

Proof. First, assume that (y^*, w^*) is a basic optimal solution of $R_\varepsilon(x^*, d^*)$.

There exists a basic matrix $E \in \mathbb{Z}^{T \times T}$ of L and basic vectors y_E^*, w_E^*, σ_E^* such that: $E\, (y_E^*\ w_E^*\ \sigma_E^*)^{tr} = d^* - Ax^*$. The matrix $E = (e_{kj})$ is invertible, and $E^{-1} = \frac{1}{\det(E)} adj(E)$, where $adj(E)$ is the adjugate matrix of E. Therefore $(y_E^*\ w_E^*\ \sigma_E^*)^{tr} = \frac{1}{\det(E)} adj(E)(d^* - Ax^*)$, and $0 \leq w_t^* = \frac{1}{\det(E)} (adj(E)(d^* - Ax^*))_{t'} \leq \varepsilon$, where w_t^* is a basic variable and where t' is the associated index in $(y_E^*, w_E^*, \sigma_E^*)$. Thus

$$|adj(E)(d^* - Ax^*))_{t'}| \leq \varepsilon |\det(E)|. \tag{9}$$

Hadamard's inequality states that $|\det(E)| \leq \prod_{j=1}^{rank(E)} \sqrt{\sum_{k=1}^{rank(E)} e_{kj}^2}$.

Thus we have $|\det(E)| \leq (l_M)^T T^{T/2}$. If $\varepsilon < \frac{1}{(l_M)^T T^{T/2}}$, then according to (9), $|(adj(E)(d^* - Ax^*))_{t'}| < 1$. Since $|(adj(E)(d^* - Ax^*))_{t'}| \in \mathbb{N}$, we have $|(adj(E)(d^* - Ax^*))_{t'}| = 0$, therefore $w_t = 0$ for any basic variable w_t and thus $w_t = 0$ for all $t = 1, ..., T$.

Thus, with $\varepsilon < \frac{1}{(l_M)^T T^{T/2}}$, if (y^*, w^*) is a basic optimal solution of $R_\varepsilon(x^*, d^*)$ then $w^* = 0$.

Now, assume that (y^*, w^*) is a non basic optimal solution of $R_\varepsilon(x^*, d^*)$. We can write (y^*, w^*) as the sum of a convex combination of basic optimal solutions (satisfying $w^{s*} = 0$ for all s) and a positive combination of extreme rays of the constraints polyhedron of $R_\varepsilon(x^*, d^*)$. Since all the optimal solutions of $R_\varepsilon(x^*, d^*)$ satisfy $w^{s*} \leq \varepsilon$ for all $s = 1, ..., S$, no extreme ray in the decomposition can have $w_t^{s*} \neq 0$ for some (s, t) with $s \in \{1, ..., S\}$ and $t \in \{1, ..., T\}$.

Finally, for any optimal solution (x^*, y^*, w^*, s^*) of PR_ε we have $w^* = 0$, and then (x^*, d^*, y^*) is a feasible solution of PR. Then from Proposition 3, $v(PR) = v(PR_\varepsilon)$ and so (x^*, d^*, y^*) is an optimal solution of PR. $\square$

To summarize, we solve PR by solving PR_ε: if the optimal solution (x^*, y^*, w^*) of PR_ε verifies $w^* = 0$ then (x^*, y^*) is an optimal solution of PR. Otherwise $w_t^* > \varepsilon$ for some t, $v(PR_\varepsilon) > M$ and PR has no solution.

6.2 Application to a production problem

In this section, we test our approach on a production problem. A company decides to install factories to manufacture several products. There are p possible factory locations and a factory at site i produces a quantity a_{it} of each product t, $t = 1, ..., T$, for a total production cost equal to α_i, $i = 1, ..., p$. The demand of a product t is uncertain and is denoted as before by $d_t = \bar{d}_t + \delta_t \Delta_t$, $t = 1, ..., T$. If the production is not sufficient, the company must buy the missing product at a high cost to meet the demand but there is a given bound to product t available on the market: β_t is the unit purchasing cost for product t, and K_t the maximal quantity of t which can be bought, $t = 1, ..., T$. The decision variables are denoted by x_i, $x_i = 1$ if a factory is installed at site i, $x_i = 0$ otherwise, $i = 1, ..., p$. The recourse variables are denoted by y_t, $t = 1, ..., T$, y_t is the (possible) lacking quantity of product t to purchase. We don't take into account the selling price of products and profit in our model. The problem to solve is the following:

$$PR\left|\begin{array}{lll}\min\limits_{x} \ \sum_{i=1}^{p}\alpha_i x_i \ + \ \max\limits_{d\in\mathfrak{D}} \min\limits_{y} \sum_{t=1}^{T}\beta_t y_t & & \\ & y_t \geq d_t - \sum_{i=1}^{p} a_{it}x_i & \forall t=1,...,T \\ & y_t \leq K_t & \forall t=1,...,T \\ & y\in\mathbb{R}_+^T & \\ x_i\in\{0,1\} \ \ i=1,...,p. & & \end{array}\right.$$

We assume that if there are factories on the p sites then the demand is always met, i.e. $\sum_{i=1}^{p} a_{it} + K_t \geq \bar{d}_t + \Delta_t$ for all $t = 1,...,T$. In that way, the problem always has a solution. In addition, the data verify $a_{it} + K_t < \bar{d}_t$, for at least one $(i,t) \in \{1,...,p\}\times\{1,...,T\}$. In this way the full recourse property is not verified since for some values of the decision variables such that $x_{\hat{i}} = 1$ and $x_i = 0$ for all $i \neq \hat{i}$ there is no solution to the recourse problem whatever the values of δ_t.

To ensure that $w_t = 0$ for all t at the end of the algorithm, we set $M = \sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{T}\beta_j K_j$ and $\varepsilon = 1$. So M is greater than the worst possible cost and ε is equal to the absolute value of any determinant of a submatrix of the constraint matrix in the recourse problem which is totally unimodular in this case (we don't need to use Hadamard's inequality in inequation (9)).

The data are generated in the following way: For a given p and a given T, we generate randomly the coefficients $\alpha, \beta, \bar{d}, K$, between 0 and 100. Then for each $t = 1,...,T$ we generate randomly Δ_t between 0 and $\bar{d}_t$. Then, we compute randomly a_{it} to ensure that $a_{it} + K_t < \bar{d}_t$, for at least one $(i,t) \in \{1,...,p\}\times\{1,...,T\}$. Finally we generate randomly $\bar{\delta}_t$ between 0 and T.

As might be expected, applying Algorithm 1 without the artificial variables (i.e. with $M = 0$) gives no solution: the program returns "Dual infeasible due to empty colomn". Eight values of (p,T) are tested. For each value, we generate five instances and we give average results in Table 1. The column 'CPU time' gives the time in seconds to obtain an optimal solution on a Bi-pro Intel Nehalem XEON 5570 at 2.93 GHz with 24 Go of RAM. The next column gives the total number of iterations in Algorithm 1. The last column precises the number of iterations where the recourse property is not satisfied, that is the number of iterations where the solution returned by the recourse problem is not a feasible solution (i.e. $w > 0$).

p	T	CPU time (s)	# iterations	# "non feasible" iterations
100	100	10	32	18
100	1500	754	95	65
300	500	735	87	44
500	300	317	56	39
500	500	2362	117	67
500	800	3480	114	80
600	700	4989	139	98
800	100	118	46	25
1500	100	714	89	41

Table 1: Average CPU time and number of iterations for p sites and T products

The most difficult instances are those where the number of sites and products are similar while it is easy to solve instances with up to 1500 sites (resp. products) and 100 products (resp. sites). The CPU time seems to be unrelated to the ratio between the number of iterations and the number of unfeasible solutions in the recourse problem.

Finally, we notice that our method to solve robust problems which does not satisfy the full recourse property is practicable and allow large scale instances to be solved.

Annex 1: problem including continuous decision variables and not verifying the full recourse property

Now we consider the problem with mixed integer decision variables: $x_i \in \mathbb{N},\ i = 1, ..., p_1,\ x_i \in \mathbb{R}_+,\ i = (p_1 + 1), ..., p$. We assume that PR has a solution and so there is M such that $v(PR) \leq M$. Following the ideas of Section 6.1 for integer decision variables, we are going to show that there is ε such that we can solve PR by solving PR_ε.

From Proposition 1 the worst scenario for any feasible x is an extreme point of $\mathfrak{D}$. Let $\{d^1, ..., d^S\}$, be the set of extreme points of $\mathfrak{D}$; all these points are integers and $S = \binom{T}{\bar{\delta}} + 1$. We can rewrite the problem PR_ε as the following MILP where

y^s and w^s correspond to the recourse variables associated to scenario s:

$$PR'_\varepsilon \left| \begin{array}{ll} \min\limits_{\substack{x,z \\ y^1,\dots,y^S \\ w^1,\dots,w^S}} & \alpha_1 x^1 + \alpha_2 x^2 + z \\ & z \geq \beta y^s + \dfrac{M}{\varepsilon}\displaystyle\sum_{t=1}^{T} w_t^s,\ s = 1,\dots,S \quad (10) \\ & A_1 x^1 + A_2 x^2 + B y^s + w^s \geq d^s,\ s = 1,\dots,S \quad (11) \\ & C_1 x^1 + C_2 x^2 \geq b \\ & x^1 \in \mathbb{N}^{p_1},\ x^2 \in \mathbb{R}_+^{p-p_1} \quad (12) \\ & y^s \in \mathbb{R}_+^q,\ w^s \in \mathbb{R}_+^T,\ s = 1,\dots,S \quad (13) \end{array} \right.$$

where $x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$, $A = (A_1, A_2)$ and $C = (C_1, C_2)$. Let $(x^*, z^*, y^{1*}, \dots, y^{S*}, w^{1*}, \dots, w^{S*})$ be an optimal solution of this program; there is $\hat{s} \in \{1, \dots, S\}$ such that constraint (10) is saturated: $z^* = \beta y^{\hat{s}*} + \frac{M}{\varepsilon}\sum_{t=1}^{T} w_t^{\hat{s}*}$ and $\hat{s}$ is the worst scenario associated to x^*.

The scheme of the proof is the following: first we prove that for ε small enough we have $w^s = 0$ for all $s = 1, \dots, S$ in any optimal solution of PR'_ε, then we deduce that $w = 0$ in any optimal solution of PR_ε and finally, from an optimal solution of PR_ε we obtain an optimal solution of PR.

Proposition 5. *If* $\varepsilon < \frac{1}{l_M^{(ST+n)}(ST+n)^{(ST+n)/2}}$ *then any optimal solution of* PR'_ε *verifies* $w^s = 0$*, for all* $s = 1, \dots, S$.

Proof. Let $(x^{1*}, x^{2*}, z^*, y^{1*}, \dots, y^{S*}, w^{1*}, \dots, w^{S*})$ be an optimal solution of PR'_ε. From constraints (10) and Proposition 3, which is true for mixed integer variables, we have for all $s = 1, \dots, S$:

$$\alpha_1 x^{1*} + \alpha_2 x^{2*} + \beta y^{s*} + \frac{M}{\varepsilon}\sum_{t=1}^{T} w_t^{s*} \leq \alpha_1 x^{1*} + \alpha_2 x^{2*} + z^* \leq M.$$

Therefore, since α and β are positive, $\frac{M}{\varepsilon}\sum_{t=1}^{T} w_t^{s*} \leq M$ and

$$w_t^{s*} \leq \varepsilon,\ \forall t, s. \quad (14)$$

Let us notice that $(x^{2*}, z^*, y^{1*}, \dots, y^{S*}, w^{1*}, \dots, w^{S*})$ is an optimal solution of

$PR'_\varepsilon(x^{1*})$. Dualizing Constraints (10) in $PR'_\varepsilon(x^{1*})$, we obtain

$$D'_\varepsilon(x^{1*}) \left| \begin{array}{ll} \max\limits_{\lambda\geq 0} \min\limits_{\substack{x^2,z \\ y^1,\dots,y^S \\ w^1,\dots,w^S}} & \alpha_2 x^2 + z + \sum_{s=1}^{S} \lambda_s \left(\beta y^s + \frac{M}{\varepsilon} \sum_{t=1}^{T} w_t^s - z \right) \\ & A_2x^2 + By^s + w^s \geq d^s - A_1x^{1*},\ s = 1, ..., S \\ & C_2x^2 \geq b - C_1x^{1*} \\ & x^2 \in \mathbb{R}_+^{p-p_1} \\ & y^s \in \mathbb{R}_+^q,\ w^s \in \mathbb{R}_+^T,\ s = 1, ..., S \end{array} \right.$$

which verifies $v(D'_\varepsilon(x^{1*})) = v(PR'_\varepsilon(x^{1*})) = v(PR'_\varepsilon)$.

Let $(\lambda', x^{2'}, z', y^{1'}, ..., y^{S'}, w^{1'}, ..., w^{S'})$ be an optimal solution of $D'_\varepsilon(x^{1*})$, then $(x^{2'}, z', y^{1'}, ..., y^{S'}, w^{1'}, ..., w^{S'})$ is an optimal solution of $D'_\varepsilon(x^{1*}, \lambda')$ which is the minimization part of $D'_\varepsilon(x^{1*})$ obtained by fixing λ to λ'.

Considering the slack variable e^s, $s = 1, ..., S$ and f, we can rewrite $D'_\varepsilon(x^{1*}, \lambda')$ as

$$D'_\varepsilon(x^{1*}, \lambda') \left| \begin{array}{ll} \min\limits_{\substack{x^2,z \\ y^1,\dots,y^S \\ w^1,\dots,w^S \\ e^1,\dots,e^S,f}} & \alpha_2 x^2 + z + \sum_{s=1}^{S} \lambda'_s \left(\beta y^s + \frac{M}{\varepsilon} \sum_{t=1}^{T} w_t^s - z \right) \\ & A_2x^2 + By^s + w^s - e^s = d^s - A_1x^{1*},\ s = 1, ..., S \\ & C_2x^2 - f = b - C_1x^{1*} \\ & x^2 \in \mathbb{R}_+^{p-p_1} \\ & y^s \in \mathbb{R}_+^q,\ w^s, e^s \in \mathbb{R}_+^T,\ s = 1, ..., S. \end{array} \right.$$

Now we are going to prove that if $\varepsilon < \frac{1}{l_M^{(ST+n)}(ST+n)^{(ST+n)/2}}$, every optimal solution $u' = (x^{2'}, z', y^{1'}, ..., y^{S'}, w^{1'}, ..., w^{S'}, e^{1'}, ..., e^{S'}, f)$, of $D'_\varepsilon(x^{1*}, \lambda')$ satisfies $w^{s\prime} = 0,\ s = 1, ..., S$. Then we shall be able to conclude the proof for PR'_ε.

We can rewrite the constraint of $D'_\varepsilon(x^{1*}, \lambda')$ as $Lu = v$, where L is the following integer matrix of rank $ST + n$,

$$L = \begin{pmatrix} A_2 & B & I_T & -I_T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 & B & I_T & -I_T & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ A_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & B & I_T & -I_T & 0 \\ C_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -I_n \end{pmatrix},$$

$$u^{tr} = (x^2, y^1, w^1, e_1, ..., y^S, w^S, e_S, f)$$

and

$$v^{tr} = (d^1 - A_1 x^{1*}, ..., d^S - A_1 x^{1*}, b - C_1 x^{1*}).$$

First, assume that u' is a basic optimal solution of $D'_\varepsilon(x^{1*}, \lambda')$. There is a $(ST+n) \times (ST+n)$-basic matrix $E = (e_{ij})$ such that

$$u' = \frac{1}{\det(E)} adj(E) v.$$

Then, for any $(t,s), t \in \{1, ..., T\}$ and $s \in \{1, ..., S\}$, there is τ such that $w_t^{s\prime} = \frac{1}{\det(E)}(adj(E)\, v)_\tau$. If $(x^{2'}, z', y^{1'}, ..., y^{S'}, w^{1'}, ..., w^{S'})$ is an optimal solution of $D'_\varepsilon(x^{1*}, \lambda')$, then $(x^{1*}, x^{2'}, z', y^{1'}, ..., y^{S'}, w^{1'}, ..., w^{S'})$ is an optimal solution of PR'_ε and, from (14) we have $w_t^{s\prime} \leq \varepsilon$, and so $|(adj(E)\, v)_\tau| \leq \varepsilon |\det(E)|$.

As in Section 6.1, let $l_M = \max_{i,j} |l_{ij}|$. According to Hadamard's inequality, we have

$$|(adj(E)\, v)_\tau| \leq \varepsilon l_M^{(ST+n)} (ST+n)^{(ST+n)/2}.$$

If we fix $\varepsilon < \frac{1}{l_M^{(ST+n)} (ST+n)^{(ST+n)/2}}$, then $|(adj(E)\, v)_\tau| < 1$. Since all the coefficients of E and v are integers, $|(adj(E)\, v)_\tau| \in \mathbb{N}$, therefore $|(adj(E)\, v)_\tau| = 0$ and $w_t^{s\prime} = 0, \forall t = 1, ..., T$.

Assume now that u' is a non basic optimal solution of $D'_\varepsilon(x^{1*}, \lambda')$. By using the same arguments as at the end of proof of Proposition 4 we can prove that no extreme ray in the u' decomposition can have $w_t^{s\prime} \neq 0$ for some (s,t) with $s \in \{1, ..., S\}$ and $t \in \{1, ..., T\}$.

Therefore, in any optimal solution of $D'_\varepsilon(x^{1*}, \lambda')$ we have:

$$w^{s\prime} = 0, \ \forall s = 1, ..., S.$$

Finally, consider an optimal solution $(x^{1*}, x^{2*}, z^*, y^{1*}, ..., y^{S*}, w^{1*}, ..., w^{S*})$ of PR'_ε. There is λ^* such that $(\lambda^*, x^{2*}, z^*, y^{1*}, ..., y^{S*}, w^{1*}, ..., w^{S*})$ is an optimal solution of $D'_\varepsilon(x^{1*})$ and thus $(x^{2*}, z^*, y^{1*}, ..., y^{S*}, w^{1*}, ..., w^{S*})$ is an optimal solution of $D'_\varepsilon(x^{1*}, \lambda^*)$, and thus $w^{s*} = 0$ for all $s = 1, ..., S$ and the proof is completed.

□

Proposition 6. *Let $\varepsilon < \frac{1}{l_M^{(ST+n)} (ST+n)^{(ST+n)/2}}$. If (x^*, d^*, y^*, w^*) is an optimal solution of PR_ε then $w^* = 0$ and (x^*, d^*, y^*) is an optimal solution of PR.*

Proof. Assume that $\varepsilon < \frac{1}{l_M^{(ST+n)}(ST+n)^{(ST+n)/2}}$. From Proposition 5, $w^s = 0$ for all $s = 1, ..., S$ in any optimal solution of PR'_ϵ. Then $w = 0$ in any optimal solution of PR_ε. Indeed, assume there is an optimal solution (x^*, d^*, y^*, w^*) of PR_ε such that $w^* > 0$; then there is an optimal solution of PR'_ε with $x = x^*$ and $w^s > 0$ for s corresponding to the worst scenario associated to x^*: a contradiction.

Now, let (x^*, d^*, y^*, w^*) be an optimal solution of PR_ε. Since $w^* = 0$, (x^*, d^*, y^*) is a solution of PR with value $v(PR_\varepsilon)$. Since, from Proposition 3, $v(PR_\varepsilon) \leq v(PR)$, (x^*, d^*, y^*) is an optimal solution of PR.

□

Annex 2: a general bound M of PR

We give here a general bound for PR, i.e. a value M which is greater than an optimal value of the robust problem if this problem has a solution. This theoretical bound can be easily improved according to each specific problem.

Assume that PR has at least one feasible x. We show how to compute M such that $v(PR) \leq M$. As in Annex 1, we can rewrite PR as the following linear program :

$$PR \left| \begin{array}{lll} \min\limits_{\substack{x,z \\ y^1,...,y^S}} & \alpha.x + z & \\ & z \geq \beta.y^s,\ s = 1, ..., S & (15) \\ & Ax + By^s \geq d^s,\ s = 1, ..., S & (16) \\ & Cx \geq b & \\ & x_i \in \mathbb{N},\ i = 1, ..., p_1,\ x_j \in \mathbb{R}_+,\ j = p_1 + 1, ..., p & \\ & y^s \in \mathbb{R}^q_+,\ s = 1, ..., S & \end{array} \right.$$

Adding the slack variables, $e^s, f^s,\ s = 1, ...S$ and g, we can rewrite the constraints of the program above as $Lu = v$ with

$$L = \begin{pmatrix} 0 & 1 & -\beta & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\beta & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\beta & -1 & 0 & 0 \\ A & 0 & B & 0 & -I_T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 & 0 & B & 0 & -I_T & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \\ A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & B & 0 & -I_T & 0 \\ C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -I_n \end{pmatrix},$$

where we assume w.l.o.g. that all the coefficients are integer (see Section 6.1),

$$u^{tr} = (x, z, y^1, e^1, f^1, ..., y^S, e^S, f^S, g)$$

and

$$v^{tr} = (0, ..., 0, d^1, ..., d^S, b).$$

Consider the relaxed problem $(\bar{PR})$ where the variables, denoted by $\bar{u}$, are in $\mathbb{R}_+$. Let $\bar{u}^*$ be an extreme point of $(\bar{PR})$. Then there exists a basic $(S + ST + n) \times (S + ST + n)$-matrix L_B of L and a basic vector $\bar{u}^*_B$ such that

$$\bar{u}^*_B = \frac{1}{\det(L_B)} adj(L_B)v.$$

Since all the coefficients of L are integers, $|\det(L_B)| \geq 1$, therefore

$$\|\bar{u}^*_B\|_\infty \leq (S + ST + n)\|adj(L_B)\|_\infty \|v\|_\infty.$$

Furthermore, since all the coefficients of $adj(L_B)$ are determinants of $(S + ST + n - 1) \times (S + ST + n - 1)$-matrices, we have, according to Hadamard's inequality:

$$\|adj(L_B)\|_\infty \leq \|M\|_\infty^{S+ST+n-1}(S + ST + n - 1)^{\frac{S+ST+n-1}{2}}.$$

Knowing that $S \leq \binom{T}{\bar{\delta}} + 1$, $\|M\|_\infty = \max(\|A\|_\infty, \|B\|_\infty, \|C\|_\infty, \|\beta\|_\infty)$ and that $\|v\|_\infty = \max(\|\bar{d}\|_\infty + \|\Delta\|_\infty, \|b\|_\infty)$, we have

$$\alpha\bar{x}^* + \bar{z}^* \leq p\|\alpha\|_\infty \|\bar{u}^*\|_\infty + \|\bar{u}^*\|_\infty,$$

and therefore

$$\alpha\bar{x}^* + \bar{z}^* \leq (p\|\alpha\|_\infty + 1)(S+ST+n)\|M\|_\infty^{S+ST+n-1}(S+ST+n-1)^{\frac{S+ST+n-1}{2}}\|v\|_\infty.$$

However according to theorem 5.6 from [**?**, chap. 5.1], we can choose a feasible u for the robust problem such that $\|u - \bar{u}^*\|_\infty \leq (S + ST + n)\Xi(L)$, where $\Xi(L)$ is the maximum over all the sub-determinants of L, and which can be bound according to Hadamard 's inequality.
Then $v(PR) \leq M = (p\|\alpha\|_\infty + 1)\|u\|_\infty \leq (p\|\alpha\|_\infty + 1)(\|\bar{u}^*\|_\infty + \|u - \bar{u}^*\|_\infty)$.

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