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Degenerate second order mean field games systems

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We consider the following second order mean field games (MFG) system with degenerate diffusion and a local coupling

$$\begin{cases} -\partial_t \phi - A_{ij} \partial_{ij} \phi + H(x, D\phi) = f(x, m(x, t)) \\ \partial_t m - \partial_{ij} (A_{ij} m) - \operatorname{div}(m D_p H(x, D\phi)) = 0 \\ m(0) = m_0, \phi(x, T) = \phi_T(x) \end{cases}$$

where

- $A : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is symmetric and nonnegative
- the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is convex in the second variable
- the coupling $f : \mathbb{R}^d \times [0, +\infty) \rightarrow [0, +\infty)$ is increasing with respect to the second variable
- m_0 is a probability density
- $\phi_T : \mathbb{R}^d \rightarrow \mathbb{R}$ is a given function

Introduced by [Lasry and Lions](#) and by [Huang, Caines and Malhamé](#) to describe Nash equilibria in differential games with infinitely many players

The optimal control problem of a typical small player

$\phi(t, x)$ is the **value function**

Dynamics :

$$dX_s = \alpha_s ds + \Sigma(s, X_s) dB_s,$$

where (α_s) is the control, (B_s) is a d -dimensional Brownian motion and $\Sigma \Sigma^T = A$

Cost :

$$\mathbb{E} \left[\int_0^T H^*(s, X_s, -\alpha_s) + f(X_s, m(s, X_s)) ds + \phi_T(X_T) \right]$$

$\forall t \in [0, T]$ $m(t, x)$ denotes the **density** of population of small players at position x

the **optimal control** is formally given by the feedback
 $(t, x) \rightarrow -D_p H(x, D\phi(t, x))$

the second equation in (MFG) is the **Kolmogorov equation** of the process (X_s) when the small player plays in an optimal way

In the MFG systems with **uniformly parabolic diffusions** (typically $A_{ij}\partial_{ij}\phi = \Delta\phi$) the solutions are **smooth**, (at least if the coupling is nonlocal and regularizing or if it has a “small growth”)

Analysis by **PDE methods** : see Cardaliaguet, Lasry, Lions and Porretta (2012), Lasry and Lions (2006, 2007), Gomes, Pimentel and Sánchez-Morgado (2013), Lions (2011), Porretta (2013)

Analysis by **stochastic techniques** : see Carmona and Delarue (2013), Huang, Malhamé and Caines (2006)

The case of **couplings with an arbitrary growth** has been discussed in Cardaliaguet, Lasry, Lions and Porretta (2012) (for quadratic hamiltonians and smooth solutions) and in Porretta (2013) (for more general hamiltonians but weak solutions)

We consider **degenerate parabolic equations** : lack of regularity $\Rightarrow$ break down of the uniformly parabolic techniques

IDEA : use convex optimization methods from optimal transport problems (see Benamou and Brenier (2000), Cardaliaguet, Carlier and Nazaret (2012))

These techniques were already used to study first order MFG systems (i.e., $A \equiv 0$): see Cardaliaguet (2013), Cardaliaguet and Graber (2013), Graber (2013)

We show the existence and uniqueness of a weak solution for the degenerate MFG system as well as the stability of solutions with respect to perturbation of the data

Assumptions

- (H1) $f : \mathbb{T}^d \times [0, +\infty) \rightarrow \mathbb{R}$ is continuous in both variables, **increasing** wrt the second variable m , and $\exists q > 1, C_1$ s.t.

$$\frac{1}{C_1}|m|^{q-1} - C_1 \leq f(x, m) \leq C_1|m|^{q-1} + C_1 \quad \forall m \geq 0$$

Moreover $f(x, 0) = 0 \quad \forall x \in \mathbb{T}^d$

- (H2) $H : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous in both variables, **convex** and differentiable in the second variable, with $D_p H$ continuous in both variable, and $\exists r > 1, C_2 > 0$ s.t.

$$\frac{1}{rC_2}|\xi|^r - C_2 \leq H(x, \xi) \leq \frac{C_2}{r}|\xi|^r + C_2 \quad \forall (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d$$

Assumptions

(H3) $A : \mathbb{T}^d \rightarrow \mathbb{R}^{d \times d}$ is Lipschitz continuous with **symmetric nonnegative** values: $\exists C_3 > 0$ s.t.

$$|A(x) - A(y)| \leq C_3 |x - y| \quad \forall x, y \in \mathbb{T}^d, \quad \xi \in \mathbb{R}^d$$

Moreover, either **$r \geq p$ or $A \equiv 0$** , where $\frac{1}{p} + \frac{1}{q} = 1$

(H4) $\phi_T : \mathbb{T}^d \rightarrow \mathbb{R}$ is of class $\mathcal{C}^2$, while $m_0 : \mathbb{T}^d \rightarrow \mathbb{R}$ is a C^1 positive density

The **Fenchel conjugate** H^* of H wrt the second variable is continuous and

$$\frac{1}{r' C_2} |\xi|^{r'} - C_2 \leq H^*(x, \xi) \leq \frac{C_2}{r'} |\xi|^{r'} + C_2 \quad \forall (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d,$$

where r' is s.t. : $\frac{1}{r} + \frac{1}{r'} = 1$

Assumptions

(H3) $A : \mathbb{T}^d \rightarrow \mathbb{R}^{d \times d}$ is Lipschitz continuous with **symmetric nonnegative** values: $\exists C_3 > 0$ s.t.

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where r' is s.t. : $\frac{1}{r} + \frac{1}{r'} = 1$

Let

$$F(x, m) = \begin{cases} \int_0^m f(x, \tau) d\tau & \text{if } m \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Then F is continuous on $\mathbb{T}^d \times (0, +\infty)$, derivable and **strictly convex** in m and

$$\frac{1}{qC_1}|m|^q - C_1 \leq F(x, m) \leq \frac{C_1}{q}|m|^q + C_1 \quad \forall m \geq 0$$

Let F^* be the **Fenchel conjugate** of F wrt the second variable then $F^*(x, a) = 0$ for $a \leq 0$, moreover,

$$\frac{1}{pC_1}|a|^p - C_1 \leq F^*(x, a) \leq \frac{C_1}{p}|a|^p + C_1 \quad \forall a \geq 0,$$

where p is s.t. : $1/p + 1/q = 1$

Key point : integral estimates

Assume that (H2) and (H3) hold true and let ϕ satisfy

$$\begin{cases} -\partial_t \phi - A_{ij}(x) \partial_{ij} \phi + H(x, D\phi) \leq \alpha(t, x) \\ \phi(x, T) \leq \phi_T(x) \end{cases}$$

in the **sense of distribution**, with $\alpha \in L^p((0, T) \times \mathbb{T}^d)$, $\phi_T \in L^\infty(\mathbb{T}^d)$

i.e. for any nonnegative test function $\zeta \in C_c^\infty((0, T] \times \mathbb{T}^d)$,

$$\begin{aligned} - \int_{\mathbb{T}^d} \zeta(T) \phi_T + \int_0^T \int_{\mathbb{T}^d} \phi \partial_t \zeta + \langle D\zeta, AD\phi \rangle + \zeta(\partial_i A_{ij} \partial_j \phi + H(x, D\phi)) \leq \\ \int_0^T \int_{\mathbb{T}^d} \alpha \zeta \end{aligned}$$

Key point : integral estimates

Theorem

Then, if ϕ is bounded below, we have

$$\|\phi\|_{L^\infty((0,T),L^m(\mathbb{T}^d))} + \|\phi\|_{L^\gamma((0,T)\times\mathbb{T}^d)} \leq C \left(\|\alpha\|_{L^p((0,T)\times\mathbb{T}^d)} + \|\phi_T\|_{L^m(\mathbb{T}^d)} \right)$$

*where $m = \frac{d(r(p-1)+1)}{d-r(p-1)}$ and $\gamma = \frac{rp(1+d)}{d-r(p-1)}$ if $p < 1 + \frac{d}{r}$
and $m = \gamma = +\infty$ if $p > 1 + \frac{d}{r}$,*

with a constant C depending on T, p, d, r, C_2, C_3 and on $\|\alpha\|_{L^p((0,T)\times\mathbb{T}^d)}, \|\phi_T\|_{L^m(\mathbb{T}^d)}$ and $\|\phi_-\|_\infty$.

When $r > 2$ and $p > 1 + d/r$, Cardaliaguet and Silvestre 2012 proved that the solution ϕ is locally Hölder continuous, with Hölder modulus depending only on the growth condition of H , the L^∞ bound on u and the L^p norm of α

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Optimal control problems in duality

An optimal control problem for a **backward HJ equation** :

the state variable ϕ is controlled by a distributed control $\alpha : (0, T) \times \mathbb{T}^d \rightarrow \mathbb{R}$ in order to minimize the criterium

$$\int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha(t, x)) \, dx dt - \int_{\mathbb{T}^d} \phi(0, x) dm_0(x)$$

when the state ϕ is driven by

$$\begin{cases} -\partial_t \phi - A_{ij}(x) \partial_{ij} \phi + H(x, D\phi) = \alpha(t, x) \\ \phi(x, T) = \phi_T(x) \end{cases}$$

Precisely :

$$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi)$$

where $\mathcal{K}_0 := \{\phi \in \mathcal{C}^2([0, T] \times \mathbb{T}^d) \mid \phi(T, x) = \phi_T(x)\}$ and

$$\begin{aligned} \mathcal{A}(\phi) = & \int_0^T \int_{\mathbb{T}^d} F^*(x, -\partial_t \phi(t, x) - A_{ij} \partial_{ij} \phi + H(x, D\phi(t, x))) \, dx dt \\ & - \int_{\mathbb{T}^d} \phi(0, x) dm_0(x) \end{aligned}$$

The second optimal control problem is an optimal control problem for a **Fokker-Plank equation** :

we control the state variable m through a vector field $v : (0, T) \times \mathbb{T}^d \rightarrow \mathbb{R}^d$ in order to minimize

$$\int_0^T \int_{\mathbb{T}^d} m(t, x) H^*(x, -v(t, x)) + F(x, m(t, x)) \, dx dt + \int_{\mathbb{T}^d} \phi_T(x) m(T, x) \, dx$$

when m solves the Fokker-Plank equation

$$\partial_t m - \partial_{ij}(A_{ij}(x)m) + \operatorname{div}(mv) = 0$$

Precisely :

$$\inf_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m, w)$$

where

$$\mathcal{K}_1 := \left\{ (m, w) \in L^1((0, T) \times \mathbb{T}^d) \times L^1((0, T) \times \mathbb{T}^d, \mathbb{R}^d) \mid \right. \\ \left. m(t, x) \geq 0 \text{ a.e.}, \int_{\mathbb{T}^d} m(t, x) dx = 1 \text{ for a.e. } t \in (0, T), \text{ and} \right. \\ \left. \partial_t m - \partial_{ij}(A_{ij}(x)m) + \operatorname{div}(w) = 0 \text{ in } (0, T) \times \mathbb{T}^d, \quad m(0) = m_0 \right. \\ \left. \text{in the sense of distribution} \right\}$$

$$\mathcal{B}(m, w) = \int_0^T \int_{\mathbb{T}^d} m(t, x) H^* \left(x, -\frac{w(t, x)}{m(t, x)} \right) + F(x, m(t, x)) \, dx dt \\ + \int_{\mathbb{T}^d} \phi_T(x) m(T, x) dx$$

$$\text{for } m(t, x) = 0, \quad m(t, x) H^* \left(x, -\frac{w(t, x)}{m(t, x)} \right) = \begin{cases} +\infty & \text{if } w(t, x) \neq 0 \\ 0 & \text{if } w(t, x) = 0 \end{cases}$$

Lemma

We have

$$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = - \min_{(m, w) \in \mathcal{K}_1} \mathcal{B}(m, w)$$

Moreover, the minimum in the right-hand side is achieved by a **unique** pair $(\bar{m}, \bar{w}) \in \mathcal{K}_1$ satisfying

$$(\bar{m}, \bar{w}) \in L^q((0, T) \times \mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}((0, T) \times \mathbb{T}^d)$$

IDEA : Use the Fenchel-Rockafellar duality Theorem

Relaxation

Let

$$\mathcal{K} := \left\{ (\phi, \alpha) \in L^\gamma((0, T) \times \mathbb{T}^d) \times L^p((0, T) \times \mathbb{T}^d) \mid D\phi \in L^r((0, T) \times \mathbb{T}^d) \right. \\ \left. \text{and which satisfies in the sense of distribution} \right. \\ \left. -\partial_t \phi - A_{ij}(x) \partial_{ij} \phi + H(x, D\phi) \leq \alpha, \quad \phi(T, \cdot) \leq \phi_T \right\}$$

Due to the presence of second order derivatives we do not expect the function ϕ to be BV (as in the first order case)

What about **trace properties** ? ϕ has a trace in a weak sense

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$. Then, for any Lipschitz continuous map $\zeta : \mathbb{T}^d \rightarrow \mathbb{R}$, the map $t \rightarrow \int_{\mathbb{T}^d} \zeta(x) \phi(t, x) dx$ has a **BV representative** on $[0, T]$.

Moreover, if we denote by $\int_{\mathbb{T}^d} \zeta(x) \phi(t^+, x) dx$ its right limit at $t \in [0, T]$, then the map $\zeta \rightarrow \int_{\mathbb{T}^d} \zeta(x) \phi(t^+, x) dx$ is continuous in $L^{m'}(\mathbb{T}^d)$.

$\Rightarrow$ for any nonnegative C^1 map $\zeta : [0, T] \times \mathbb{T}^d \rightarrow \mathbb{R}$, one can write the integration by part formula: for any $0 \leq t_1 \leq t_2 \leq T$,

$$\begin{aligned}
 - \left[\int_{\mathbb{T}^d} \zeta \phi \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \phi \partial_t \zeta + \langle D\zeta, A D\phi \rangle + \zeta (\partial_i A_{ij} \partial_j \phi + H(x, D\phi)) \leq \\
 \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \alpha \zeta
 \end{aligned}$$

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$. Then, for any Lipschitz continuous map $\zeta : \mathbb{T}^d \rightarrow \mathbb{R}$, the map $t \rightarrow \int_{\mathbb{T}^d} \zeta(x) \phi(t, x) dx$ has a **BV representative** on $[0, T]$.

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$\Rightarrow$ for any nonnegative C^1 map $\zeta : [0, T] \times \mathbb{T}^d \rightarrow \mathbb{R}$, one can write the integration by part formula: for any $0 \leq t_1 \leq t_2 \leq T$,

$$- \left[\int_{\mathbb{T}^d} \zeta \phi \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \phi \partial_t \zeta + \langle D\zeta, A D\phi \rangle + \zeta (\partial_i A_{ij} \partial_j \phi + H(x, D\phi)) \leq \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \alpha \zeta$$

Proof :

For any Lipschitz continuous, nonnegative map ζ ,

$$-\frac{d}{dt} \int_{\mathbb{T}^d} \zeta \phi(t) + \int_{\mathbb{T}^d} \langle D\zeta, AD\phi(t) \rangle + \zeta(\partial_i A_{ij} \partial_j \phi + H(x, D\phi) - \alpha) \leq 0,$$

in the sense of distribution

As the second integral is in $L^1((0, T))$, the map $t \rightarrow \int_{\mathbb{T}^d} \zeta \phi(t)$ is BV

If now ζ is Lipschitz continuous and changes sign, one can write

$\zeta = \zeta^+ - \zeta^-$ and the map $t \rightarrow \int_{\mathbb{T}^d} \zeta \phi(t) = \int_{\mathbb{T}^d} \zeta^+ \phi(t) - \int_{\mathbb{T}^d} \zeta^- \phi(t)$ is still BV

The continuity with respect to ζ comes from the $L^\infty((0, T), L^m(\mathbb{T}^d))$ estimate on ϕ

We extend the functional $\mathcal{A}$ to $\mathcal{K}$

$$\mathcal{A}(\phi, \alpha) = \int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha(x, t)) \, dx dt - \int_{\mathbb{T}^d} \phi(x, 0) m_0(x) \, dx \quad \forall (\phi, \alpha) \in \mathcal{K}$$

Proposition

We have

$$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = \inf_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha)$$

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$ and $(m, w) \in \mathcal{K}_1$.

Assume that $mH^*(\cdot, -w/m) \in L^1((0, T) \times \mathbb{T}^d)$ and $m \in L^q$.

Then, for a.e. $t \in [0, T]$, $m(t)\phi(t)$ is integrable and

$$\left[\int_{\mathbb{T}^d} m\phi \right]_t^T + \int_t^T \int_{\mathbb{T}^d} m \left(\alpha + H^*(x, -\frac{w}{m}) \right) \geq 0$$

and

$$\left[\int_{\mathbb{T}^d} m\phi \right]_0^t + \int_0^t \int_{\mathbb{T}^d} m \left(\alpha + H^*(x, -\frac{w}{m}) \right) \geq 0.$$

Moreover, if equality holds in the first inequality for $t = 0$, then $w = -mD_p H(x, D\phi)$ a.e.

Proof :

Inequality $\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) \geq \inf_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha)$ holds obviously

Let $(\phi, \alpha) \in \mathcal{K}$, then $\forall (m, w) \in \mathcal{K}_1$ with $mH^*(\cdot, -w/m) \in L^1$ we have,

$$\begin{aligned} \mathcal{A}(\phi, \alpha) &= \int_0^T \int_{\mathbb{T}^d} F^*(\alpha) - \int_{\mathbb{T}^d} \phi(0) m_0 \\ &\geq \int_0^T \int_{\mathbb{T}^d} \alpha m - F(m) - \int_{\mathbb{T}^d} m_0 \phi(0) \\ &\geq \int_0^T \int_{\mathbb{T}^d} -mH^*(x, -\frac{w}{m}) - F(m) - \int_{\mathbb{T}^d} m(T) \phi_T = -\mathcal{B}(m, w) \end{aligned}$$

Taking the sup with respect to (m, w) in the right-hand side

$$\mathcal{A}(\phi, \alpha) \geq - \inf_{(m, w) \in \mathcal{K}_1} \mathcal{B}(m, w) = \inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi)$$

Proposition

*The relaxed problem has **at least one solution** $(\bar{\phi}, \bar{\alpha}) \in \mathcal{K}$ which is bounded below by a constant depending on $\|\phi_T\|_{C^2}$, on $\|A_{ij}\|_{C^0}$ and on $\|H(\cdot, D\phi_T)\|_{\infty}$.*

Therefore, thanks to the equivalence, we have a infimum for $\mathcal{A}$ over $\mathcal{K}_0$ and a minimum for $\mathcal{B}$ over $\mathcal{K}_1$

$$\min_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha) = \inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = - \min_{(m, w) \in \mathcal{K}_1} \mathcal{B}(m, w)$$

Existence and uniqueness

We say that a pair $(\phi, m) \in L^\gamma((0, T) \times \mathbb{T}^d) \times L^q((0, T) \times \mathbb{T}^d)$ is a **weak solution to (MFG)** if

(i) $D\phi \in L^r$, $mH^*(\cdot, D_p H(\cdot, D\phi)) \in L^1$ and $mD_p H(\cdot, D\phi) \in L^1$

(ii) The following inequality holds in the sense of distribution

$$-\partial_t \phi - \partial_i (A_{ij}(x) \partial_j \phi) + (\partial_i A_{ij}) \partial_j \phi + H(x, D\phi) \leq f(x, m) \quad \text{in } (0, T) \times \mathbb{T}^d,$$

with $\phi(T, \cdot) \leq \phi_T$

(iii) The following equation holds in the sense of distribution

$$\partial_t m - \partial_{ij} (A_{ij}(x) m) - \operatorname{div}(m D_p H(x, D\phi)) = 0 \quad \text{in } (0, T) \times \mathbb{T}^d, \quad m(0) = m_0$$

(iv) The following equality holds :

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} m(t, x) (f(x, m(t, x)) + H^*(x, D_p H(x, D\phi)(t, x))) \, dx dt \\ & + \int_{\mathbb{T}^d} m(T, x) \phi_T(x) - m_0(x) \phi(0, x) \, dx = 0 \end{aligned}$$

Theorem

There **exists** a weak solution $(\bar{\phi}, \bar{m})$ to (MFG).

Moreover this solution is **unique** in the following sense: if $(\bar{\phi}, \bar{m})$ and $(\bar{\phi}', \bar{m}')$ are two solutions, then $\bar{m} = \bar{m}'$ a.e. and $\bar{\phi} = \bar{\phi}'$ in $\{\bar{m} > 0\}$.

Finally, there exists a solution which is **bounded below** by a constant depending on $\|\phi_T\|_{C^2}$, on $\|A_{ij}\|_{C^0}$ and on $\|H(\cdot, D\phi_T)\|_{\infty}$.

Proof :

Let $(\bar{m}, \bar{w}) \in \mathcal{K}_1$ be a minimizer of $\min_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m, w)$
 and $(\bar{\phi}, \bar{\alpha}) \in \mathcal{K}$ be a minimizer of $\min_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha)$

We have $-\mathcal{B}(\bar{m}, \bar{w}) = \mathcal{A}(\bar{\phi}, \bar{\alpha})$

$$\int_0^T \int_{\mathbb{T}^d} F^*(x, \bar{\alpha}) + F(x, \bar{m}) + \bar{m} H^* \left(x, -\frac{\bar{w}}{\bar{m}} \right) dx dt + \int_{\mathbb{T}^d} \phi_T \bar{m}(T) - \bar{\phi}(0) m_0 dx = 0$$

By convexity of F ,

$$F^*(x, \bar{\alpha}) + F(x, \bar{m}) - \bar{\alpha} \bar{m} \geq 0,$$

hence

$$\int_0^T \int_{\mathbb{T}^d} \bar{\alpha} \bar{m} + \bar{m} H^* \left(x, -\frac{\bar{w}}{\bar{m}} \right) dx dt + \int_{\mathbb{T}^d} \phi_T \bar{m}(T) - \bar{\phi}(0) m_0 dx \leq 0$$

Therefore $\bar{w} = -\bar{m} D_p H(\cdot, D\bar{\phi})$ a.e. and $\bar{\alpha}(t, x) = f(x, \bar{m}(t, x))$ a.e.

and $(\bar{\phi}, \bar{m})$ is a weak solution

Recall that :

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$ and $(m, w) \in \mathcal{K}_1$. Assume that $mH^*(\cdot, -w/m) \in L^1((0, T) \times \mathbb{T}^d)$ and $m \in L^q$.

Then, for a.e. $t \in [0, T]$, $m(t)\phi(t)$ is integrable and

$$\left[\int_{\mathbb{T}^d} m\phi \right]_t^T + \int_t^T \int_{\mathbb{T}^d} m \left(\alpha + H^*(x, -\frac{w}{m}) \right) \geq 0$$

and

$$\left[\int_{\mathbb{T}^d} m\phi \right]_0^t + \int_0^t \int_{\mathbb{T}^d} m \left(\alpha + H^*(x, -\frac{w}{m}) \right) \geq 0.$$

Moreover, if equality holds in the first inequality for $t = 0$, then $w = -mD_p H(x, D\phi)$ a.e.

Conversely, using the convexity of F and the Lemma, we can show that any weak solution $(\bar{\phi}, \bar{m})$ of (MFG) is such that :

the pair $(\bar{m}, -\bar{m}D_p H(\cdot, D\bar{\phi}))$ is the minimizer of $\min_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m, w)$

while $(\bar{\phi}, f(\cdot, \bar{m}))$ is a minimizer of $\min_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha)$

Uniqueness : Since the solution of $\min_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m, w)$ is unique $\bar{m}$ is unique

The proof of the uniqueness of $\bar{\phi}$ is more technical

Stability

Assume that (A^n) , (H^n) , (f^n) , m_0^n and ϕ_T^n satisfy conditions (H1)... (H4) uniformly with respect to n and converge to A , H , f , m_0 and ϕ_T locally uniformly.

Theorem

Let (ϕ^n, m^n) be a weak solution of (MFG) associated with A^n , H^n , f^n and with the initial and terminal conditions m_0^n and ϕ_T^n .

Assume also that the sequence ϕ^n is uniformly bounded below.

Then (m^n) converges strongly to m in L^q while ϕ^n converges weakly and up to a subsequence to a map ϕ in L^γ , where the pair (ϕ, m) is a weak solution to (MFG).

In particular the weak solution is stable with respect to **viscous approximation**

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