Reduced-order minimum time control of advection-reaction-diffusion systems via dynamic programming.

Dante Kalise, Axel Kröner

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Dante Kalise[1] and Axel Kröner[2]

***Abstract*— We present a numerical approach for the time-optimal feedback control of an advection-reaction-diffusion model. Our approach is composed by three main building blocks: approximation of the abstract system dynamics, feedback computation based on dynamic programming and state observation. For the approximation of the abstract dynamics, we consider a finite element semi-discretization in space, leading to a large-scale dynamical system, whose dimension is reduced by means of a Balanced Truncation algorithm. Next, we apply the dynamic programming principle over the reduced model, and characterize the value function of the optimal control as a viscosity solution of a Hamilton-Jacobi-Bellman equation, which is numerically approximated with a semi-Lagrangian scheme. Finally, the computation of the corresponding feedback controls and its insertion into the control loop is performed by implementing a Luenberger observer.**

I. Introduction

In this paper, we consider a minimum time problem with dynamics governed by a one-dimensional advection-reaction-diffusion model; such a problem arises, for instance, in minimum time stabilization of fluid flow passing through slender structures, or in the control of chemically reactive processes. The abstract evolutionary system is given by

$$\begin{cases} \partial_t y = \mathcal{A}y + \mathcal{B}u, \\ y(0) = x, \\ \quad z = \mathcal{C}y\,, \end{cases} \tag{I.1}$$

where $y = y(\xi, t)$ represents the state of the system (here ξ and t stands for the spatial coordinate and time, respectively), $u = u(t)$ is a time-dependent scalar control signal, and $z = z(\xi, t)$ represents an observation. The system is spatially confined to $\xi \in \Omega = (0, L) \subset \mathbb{R}$, $L > 0$, and time $t \in [0, T]$. Using the usual notations for Lebesgue and Sobolev spaces, the free-dynamics operator $\mathcal{A}\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined as

$$\mathcal{A}y = a_1 \partial_\xi^2 y + a_2 \partial_\xi y + a_3 y,$$

with $a_1 \in \mathbb{R}^+$, $a_2, a_3 \in \mathbb{R}$, whereas the control operator $\mathcal{B}$ and the observation operator $\mathcal{C}$ are both indicator functions over open subsets $\Omega_c \subset \Omega$ and $\Omega_o \subset \Omega$, respectively. For an initial state $x \in H_0^1(\Omega)$ and $u \in L^\infty(0, T)$ the system has a unique solution in $C((0, T), C(\bar{\Omega}))$, see [9, p. 360] and [21, p.19]. Furthermore, the uncontrolled system is exponentially stable for $a_3 < \pi^2 a_1$. For a target $\mathcal{T} = \{\, z \in C(\bar{\Omega}) \mid \|z\|_{C(\bar{\Omega})} \leq \varepsilon \,\}$, $\varepsilon > 0$, we address the following minimum time problem:

$$\begin{cases} \min\limits_{u \in \mathcal{U}} T, \quad \text{s.t.} \\ \quad \partial_t y = \mathcal{A}y + \mathcal{B}u\,, \\ \quad y(0) = x, \\ \quad y(T) \in \mathcal{T} \end{cases} \tag{I.2}$$

for $\mathcal{U} = L^\infty((0, T), U)$ with

$$U = \{\, u \in \mathbb{R} \mid u_a \leq u(s) \leq u_b \text{ a.e.} \,\}\,,$$

and bounds $u_a, u_b \in \mathbb{R}$. For further comments on the continuous problem for the case $a_2 = 0$, we refer to [18].

Our aim is to derive approximate controllers in feedback form for the aforementioned problem. A first step towards this goal is to perform a finite element semi-discretization in space of the advection-reaction-diffusion equation. The resulting control problem replaces the abstract dynamics (I.1) by a large-scale, finite dimensional dynamical system. At this point, it would be theoretically feasible to apply the Dynamic Programming Principle (DPP) in order to characterize the value function of the optimal control problem as the viscosity solution of a Hamilton-Jacobi-Bellman (HJB) equation, as presented in [13]. However, the fact that such an equation is defined over a state space of the same dimension as the semi-discretized dynamics, renders it computationally intractable. Therefore, it is necessary to include an additional approximation step in order to reduce the dimensionality of the semi-discrete system. A standard tool for this purpose is the application of a Balanced Truncation algorithm, which synthesizes the relevant information in terms of controllability and observability of the system, into a lower dimensional model of prescribed accuracy (with respect to a measure of the input-output mapping). Once the reduced dynamics have been obtained, the application of the DPP over the resulting system leads to an approximate, low-order feedback controller. However, the control law assumes a full knowledge of the reduced state of the system which, according to (I.1), is not available; instead, we receive a measured output z. Therefore, the final step in the control loop consists in the implementation of a reduced order Luenberger observer in order to generate a reliable estimate of the internal dynamics of the system, for an accurate computation of the feedback mapping.

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To set this paper in perspective, we relate our approach to the previous works [2], [17], [19], [16], which also consider optimal feedback control problems for infinite-dimensional dynamics, using either proper orthogonal decomposition or spectral elements to obtain a low dimensional semi-discrete system. A numerical solution for feedback control problems for nonlinear parabolic equations is considered in [7]. Numerical implementations and approximation results for feedback problems of (second-order) hyperbolic equations using Riccati equations can be found in [14], [15], and references therein. Similar approaches, based on model predictive control, which can be interpreted as a relaxed version of dynamic programming, have been presented in [8], [4]. Minimum time problems were first considered in [6]. For time optimal control problems of parabolic equations see [12], [18], [20], [23], [22], [27]. The novelty of this article resides in the combination of HJB techniques for the computation of time-optimal feedback controllers with model reduction and state observation algorithms providing thus a consistent approach for closed-loop control of infinite-dimensional systems.

The paper is organized as follows. In Section II , the semi-discretization of the dynamics and its reduction via Balanced Truncation is presented. In Section III we introduce the dynamic programming framework and a semi-Lagrangian scheme for the solution of the resulting HJB equation. In Section IV, we propose a Luenberger observer for state estimation in the closed-loop implementation. Section V reports numerical examples assessing the performance and capabilities of the proposed approach.

II. Discretization and model reduction

In this section, we present a semi-discretization of the reaction-diffusion-advection equation (I.1) resulting in a finite-dimensional dynamical system. At this level, the application of a HJB-based approach is still unfeasible from a computational point of view, since the PDE characterizing the value function of the problem is posed over a state space of the same dimension as the semi-discretization. To reduce the dimensionality of the problem, we apply a Balanced Truncation algorithm to the semi-discrete dynamical system. In this way, we obtain a low-dimensional model for which the corresponding HJB equation is numerically solvable. The building blocks of this procedure are illustrated in Figure 1. Balanced Truncation for optimal control of evolution equations has been considered in [24]. Classical Balanced Truncation is limited to linear problems, but it has desirable properties, such as stability preservation.

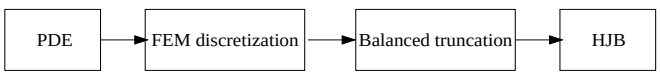

Fig. 1: Building blocks for the low-order optimal control problem.

To discretize (I.1) in space we use continuous, piecewise linear finite elements. For a given $N \in \mathbb{N}$, let $\mathcal{G} = \{ j\Delta x \}_{j=0}^{N+1}$, $\Delta x = \frac{L}{N+1}$, be a regular subdivision of the interval $[0, L]$. For the corresponding standard nodal basis functions $\{ \varphi_i \}_{i=1}^{N}$, we introduce the mass matrix $M \in \mathbb{R}^{N\times N}$ with $M_{i,j} = (\varphi_i, \varphi_j)$, the stiffness matrix $K \in \mathbb{R}^{N\times N}$ with $K_{i,j} = (\partial_\xi \varphi_i, \partial_\xi \varphi_j)$, and for the discretization of the advection term $D \in \mathbb{R}^{N\times N}$ with $D_{i,j} = (\partial_\xi \varphi_i, \varphi_j)$, where $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\Omega)$. Further, we define the observation matrix $\mathcal{C}^N = M_{N_o}^{-1} \chi^{N_o,N} M \in \mathbb{R}^{N_o\times N}$, $N_o \leq N$, for matrix $\chi^{N_o,N} \in \mathbb{R}^{N_o\times N}$ of rank N_o with entries $\chi_{i,j}^{N_o,N} \in \{ 0, 1 \}$, according to the observation set Ω_o and the mass matrix $M_{N_o} \in \mathbb{R}^{N_o\times N_o}$ which we choose according to the observed components. For the ansatz $y^{\Delta x} = \sum_{i=1}^{N} y_i \varphi_i$, where y_i are time-dependent nodal values of the finite element functions $y^{\Delta x}$, we introduce $y^N = (y_1, \ldots, y_N)^T$ which satisfies the semi-discrete system given by

$$\begin{cases} \partial_t y^N = \mathcal{A}^N y^N + \mathcal{B}^N u, \\ y^N(0) = x^N, \\ \qquad z = \mathcal{C}^N y^N \end{cases} \tag{II.1}$$

for $x^N \in \mathbb{R}^N$ with $x_i^N = (x, \varphi_i)$, and

$$\mathcal{A}^N = M^{-1} \left(a_1 K + a_2 D + a_3 M \right),$$
$$\mathcal{B}^N = M^{-1} \chi_c^N, \quad [\chi_c^N]_i = (\chi_c, \varphi_i),$$

with χ_c the characteristic function of Ω_c. Error estimates in L^2-norm for this type of semi-discrete approximation are presented in [26].

For the convenience of the reader, we shortly recall the basic idea of Balanced Truncation, see [25], [24]. In the following we assume that $\mathcal{A}^N$ is stable, $(\mathcal{A}^N, \mathcal{B}^N)$ is controllable, and $(\mathcal{A}^N, \mathcal{C}^N)$ is observable. To simplify the notation we drop the index N. First, note that under the aforementioned assumptions, there exists a controllability Gramian $\mathcal{P}$ and an observability Gramian $\mathcal{Q}$ solving the Lyapunov equations

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} = -\mathcal{C}^T C,$$
$$\mathcal{Q}\mathcal{A}^T + \mathcal{A}\mathcal{Q} = -\mathcal{B}\mathcal{B}^T.$$

Since the Gramians are positive-definite, We can compute their Cholesky factors R and L, i.e.

$$\mathcal{P} = RR^T, \quad \mathcal{Q} = LL^T,$$

and solve the singular value decomposition of $L^T R$. We fix a reduced-order model dimension $r \in \mathbb{N}$, and split

$$L^T R = (U_1\ U_2) \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix},$$

with orthonormal matrices $U = (U_1\ U_2)$ and $V = (V_1\ V_2)^T$, and diagonal matrices

$$\Sigma_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_r), \quad \Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \ldots, \sigma_l),$$

with

$$\sigma_1 \geq \cdots \geq \sigma_r \gg \sigma_{r+1} \geq \cdots \geq \sigma_l > 0,$$

and $l = \operatorname{rank}(L^T R)$. The singular values of $L^T R$, known in this context as Hankel singular values, provide a simultaneous measure of both controllability and observability energies of the corresponding balanced state. By setting

$$W = LU_1\Sigma_1^{-\frac{1}{2}}, \quad S = RV_1\Sigma_1^{-\frac{1}{2}}$$

we can compute the following transformation

$$\mathcal{A}^r = W^T \mathcal{A} S, \quad \mathcal{B}^r = S^T \mathcal{B}, \quad \mathcal{C}^r = \mathcal{C} W\,,$$

and obtain the reduced problem

$$\begin{cases} \partial_t y^r = \mathcal{A}^r y^r + \mathcal{B}^r u, \\ y^r(0) = W^T x, \\ z^r = \mathcal{C}^r y^r \end{cases} \tag{II.2}$$

with $y^r \in \mathbb{R}^r$.

For the error between the solution of the reduced model (II.2) in comparison to the solution of the full model (II.1) on $(0,\infty)$ there holds, see, e.g., [24],

$$\|z^r - z\|_{L^2(0,\infty)} \leq 2(\sigma_{r+1} + \cdots + \sigma_l)\, \|u\|_{L^2(0,\infty)}\,.$$

This estimate gives an upper bound for the error in the observation depending on the dimension of the reduced model.

III. The dynamic programming approach

By applying a Balanced Truncation algorithm over the semi-discrete model, we obtain a reduced dynamical system. This leads to the following approximated minimum time problem

$$\begin{cases} \min\limits_{u\in\mathcal{U}} T, \quad \text{s.t.} \\ \quad \partial_t y^r = \mathcal{A}^r y + \mathcal{B}^r u\,, \\ \quad y^r(0) = W^T x, \\ \quad y^r(T) \in \mathcal{T}\,, \end{cases} \tag{III.1}$$

This latter problem is solved by applying the dynamic programming principle, which in this case is given by

$$T(x) = \inf_{u\in\mathcal{U}} (\tau + T(y_x(\tau,u))) \tag{III.2}$$

for all $x \in \mathcal{R} = \{\, x \in \mathbb{R}^l \mid T(x) < \infty \,\}$, $\tau \in [0, T(x))$ with $x \notin \mathcal{T}$. Here $y_x(\tau,u)$ denotes the state at time τ for initial state x at time zero and control u. The DPP leads to the following HJB equation characterizing the minimum time

$$\begin{cases} \mathcal{H}(x, DT) = 0 & \text{in } \mathcal{R}\setminus\mathcal{T}, \\ T = 0 & \text{on } \partial\mathcal{T}, \\ T(x) \to +\infty & \text{as } x \to x_0 \in \partial\mathcal{R} \end{cases} \tag{III.3}$$

with Hamiltonian

$$\mathcal{H}(x,p) = \sup_{u\in U} (-f(x,u)^T p) - 1\,,$$

and

$$f(x,u) = \mathcal{A}^r x + \mathcal{B}^r u\,.$$

Applying the Kruzkov transform

$$v(x) = \begin{cases} 1 - e^{-T(x)} & x \in \mathcal{R} \\ 1 & x \notin \mathcal{R} \end{cases}$$

we further obtain

$$\begin{cases} v + \mathcal{H}(x, Dv) = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{T}, \\ v = 0 & \text{on } \partial\mathcal{T}. \end{cases} \tag{III.4}$$

In order to solve the HJB equation numerically, we apply a semi-Lagrangian scheme following [3]; for a general introduction to semi-Lagrangian schemes we refer to [5], [10]. Be begin by discretizing the dynamical system in time with stepsize h, and apply the DPP for the discrete-time dynamical system. Then, for spatial mesh parameter $k \in \mathbb{R}_+$, we introduce a regular mesh $\mathcal{G} = \{\, x_I \mid I \in \mathbb{Z}^r,\ Ik \in \mathcal{S} \,\}$ and denote the set of all multi-indices I with $x_I = Ik \in \mathcal{G}$ by J. We restrict the computations to a domain $\mathcal{S} \subset \mathbb{R}^r$, and impose an artificial Dirichlet boundary condition on $\partial\mathcal{S}$ which we set to the value 1. The fully discretized HJB equation then reads

$$\begin{cases} v^{h,k}(x_I) = \min\limits_{u\in U} (\beta \mathcal{I}[v^{h,k}](x_I + h f(x_I,u))) + 1 - \beta, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } I \in J_{in}, \\ v^{h,k}(x_I) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } I \in J_{\mathcal{T}}, \\ v^{h,k}(x_I) = 1 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } I \in J_{\text{out}} \end{cases} \tag{III.5}$$

with $\beta = e^{-h}$ and

$$\begin{aligned} J_{\text{out}} &= \{\, I \mid x_I + h f(x_I,u) \notin \mathcal{S} \text{ for any } u \in U \,\}\,, \\ J_{\mathcal{T}} &= \{\, I \mid x_I \in \mathcal{T} \cap \mathcal{S} \,\}\,, \\ J_{\text{in}} &= I \in J \setminus (J_{\text{out}} \cup J_{\mathcal{T}}). \end{aligned} \tag{III.6}$$

Here $\mathcal{I}[\cdot]$ denotes a linear interpolation operator. Having approximated the value function by means of the aforementioned scheme, we can recover the feedback controller for a given state x by

$$u(x) = \arg\min_{u\in U} \Big(\beta \mathcal{I}[v^{h,k}](x + h f(x,u)) \Big). \tag{III.7}$$

This latter expression requires the knowledge of the whole state, whereas our system considers an observation equation. In the next section, we introduce an observer implementation for an accurate trajectory computation.

IV. Luenberger Observer

In order to generate a coherent link between the observation and the feedback law, it is necessary to implement a state observer. More specifically, the approximation of system (I.1) generates a dynamical output z, whereas the feedback mapping (III.7) assumes that a full knowledge of the internal state y is available. By implementing a Luenberger observer, we are able to circumvent this difficulty and to close the control loop. The interplay of the observer and control blocks is illustrated in Figure 2. The observer is built upon the reduced order dynamics, corresponding to the "true" state from which the feedback mapping can be computed. The estimated state $\hat{y}$ is governed by

$$\begin{cases} \partial_t \hat{y} = \mathcal{A}^r \hat{y} + \mathcal{B}^r u + L(\mathcal{C}^r \hat{y} - z)\,, \\ \hat{y}(0) = \hat{y}_0, \end{cases}$$

where $\hat{y}_0 \in \mathbb{R}^r$ and $L \in \mathbb{R}^{r\times N_o}$ needs to be computed such that $\mathcal{A}^r - L\mathcal{C}^r$ is asymptotically stable and with a

decay rate faster than the free system dynamics $\mathcal{A}^r$. In our particular setting, we recall that $\mathcal{A}^r$ is asymptotically stable, and therefore our problem reduces to find a suitable L which accelerates the decay of the estimation error. The observer gives us now a corresponding state estimate $\hat{y}$ which is inserted in the expression (III.7), yielding a feedback $u = u(\hat{y})$ to be connected to the full-order system.

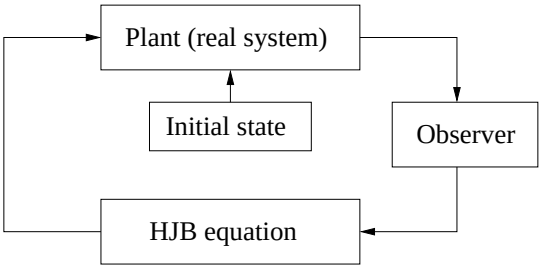

Fig. 2: Overview over the observer mechanism

V. Numerical examples

We illustrate the application of the proposed approach and the interplay between the different building blocks. First, we describe specific details concerning common settings for the numerical implementation.

System configuration and parameters. In both tests, the physical domain corresponds to $\Omega = [0, 1]$, the operator $\mathcal{B}$ is an indicator function over the subset $\Omega_c = (0.4, 0.6)$, and the observation operator $\mathcal{C}$ is an indicator function over $\Omega_o = (0.3, 0.7)$ for Test 1 and $\Omega_o = \Omega$ for Test 2. In general, the main issue when prescribing a setting relates to abstract controllability and observability assumptions, which play a role in both the reduction and in the HJB step; the aforementioned configuration has been selected in order to fulfill these requirements. Nevertheless, we report that the presented approach can be implemented also for point control/observation configurations. The control set corresponds to $U = [-1, 1]$. Simulations are shown with the initial condition $x(\xi) = 0.2\sin(\pi\xi)$, and for a time frame of $1[s]$.

Semi-discretization and model reduction. Semi-discretization in space is performed with piecewise linear finite elements. In both tests, the results were obtained with a discretization of 200 elements in space. The resulting large-scale system is reduced by means of a Balanced Truncation algorithm; we use the numerical implementation provided by the MORLAB package [1].

Hamilton-Jacobi-Bellman equation. The numerical approximation of the resulting HJB equation by means of a semi-Lagrangian schemes follows the general guidelines presented in the literature as in [5], [10]. The equation is solved in a subdomain of the reduced state-space, in our case the computational domain is given by $Q = [-1, 1]^3$ considering only three reduced states in both tests. The state-space discretization is given by the space parameter $k = 0.025$ and the pseudo-time parameter $h = k/\|\mathcal{A}^r\|$, which gives a good balance between accuracy and computation time (as arrival points are computed in close neighborhood of their departure). The semi-Lagrangian discretization leads to the system (III.5) which is solved by means of a value iteration algorithm defined upon the grid point values $V = \{v^{h,k}(x_I)\}$,

$$V^{n+1} = S(V^n)$$

$$[S(V)]_I = \begin{cases} \min_{u\in U}(\beta\mathcal{I}[V](x_I + hf(x_I, u))) + 1 - \beta, & \text{for } I \in J_{in}, \\ 0 & \text{for } I \in J_{\mathcal{T}}, \\ 1 & \text{for } I \in J_{\text{out}} \end{cases}$$

which in our case is stopped when two consecutive iterations hold $\|V^n - V^{n+1}\| \leq k^2$. The control set U is discretized into 10 equidistant points. The target set $\mathcal{T}$ is a ball around the origin of radius $\epsilon = h$.

State observer and trajectory computation. The implementation of the observer, in this particular case, reduces to determining $L \in \mathbb{R}^{r\times r}$ such that the eigenvalues of $\mathcal{A}^r - L\mathcal{C}^r$ decay sufficiently fast compared to the system dynamics. For this problem, we prescribe eigenvalues 10 times faster than the slowest eigenvalues of $\mathcal{A}^r$, and the initial observation estimate is set $\hat{y}(0) = 0$. For trajectory computation, at every time step, once the current state of the system has been estimated, the feedback rule is computed by means of (III.7), and the system advances in time with a second-order implicit integrator. Let us stress that, because of the computational effort performed in the HJB step, where the value function is computed for the whole state space of interest, the cpu time related to trajectory and feedback computation is negligible, as it mainly reduces to a single-node evaluation of arrival points.

We present two numerical tests, first a purely reaction-diffusion model, and in a further step we also include advective effects.

A. Test 1: reaction-diffusion

In this first test we set for the free-dynamics operator $a_1 = 0.05$, $a_2 = 0$, and $a_3 = 0.1$, i.e., advection is not considered. The resulting reaction-diffusion system is exponentially stable, and thus our control goal reduces to achieve faster regulation of the initial state to the origin. After discretizing the system in space by finite elements we balance the resulting large-scale system. Figure 3 shows the evolution of the Hankel singular values in balanced coordinates; the error of the truncation step is governed by the largest Hankel singular value which has been neglected, and in our case, to consider 3 reduced states leads to an acceptable error. Note that this figure also assess our choice of Balanced Truncation for the reduction procedure, as the fast decay of the singular values is fundamental if a good approximation with a few number of states is required. For a system with 3 reduced states, computation time of the HJB step is of approximately 50 seconds. CPU time increases dramatically for high-dimensional problems, however, by using accelerated iterative algorithms as in [3], it is possible

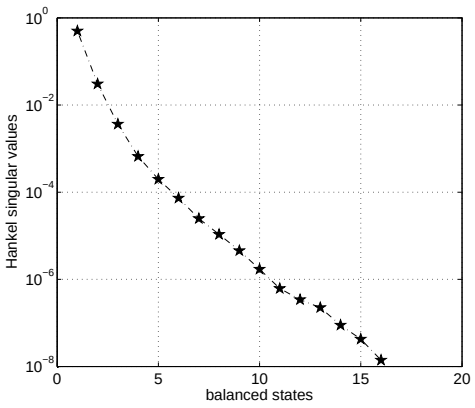

Fig. 3: Test 1 (without advection). Decay of the Hankel singular values in balanced coordinates.

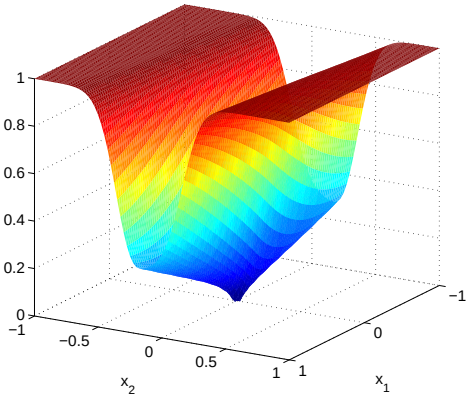

Fig. 4: Test 1 (without advection). Value function: cut at $x_3 = 0$.

to solve similar problems in four dimensions in the same amount of time, and to keep a cpu time for five dimensional problems on the order of 10 minutes (for coarse meshes).

The next step is illustrated in Figure 4, where the value function obtained via the DPP approach is shown. Such value function shape often appears in minimum time control of stable linear systems, and is related to switching curves splitting the associated control space into a discrete number of sets, i.e., a bang-bang type of control.

Dynamical response of the system is presented in Figures 5 and 6, where the performance of the minimum time controller can be observed compared to the free dynamics, the fast decay of the estimation error, and the respective control signal, which emulates the expected bang-bang behavior. Differences with an exact bang-bang controller, namely the chattering after the first switching, are observed due to the approximate character of the discrete HJB equation and the computation of trajectories over a discrete state-space grid (where switching curves are also approximated), which is a natural limitation of discrete dynamic programming-based approaches. Aiming at a more realistic online implementation, in [11], the authors have considered an ℓ-1 penalized minimum time problems where the bang-bang structure is replaced by a bang-zero-bang behavior which reduces chat-

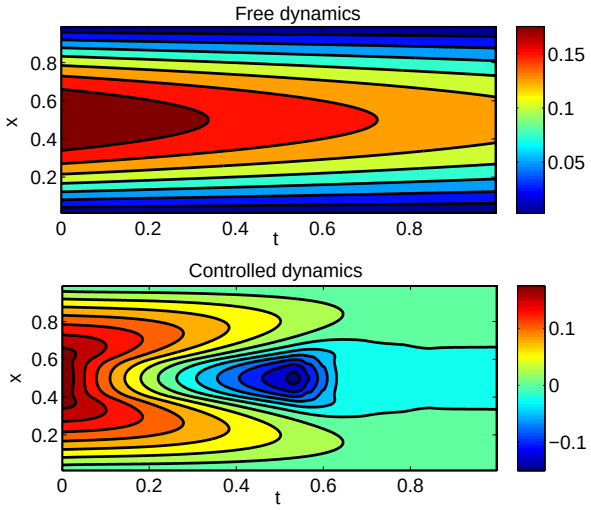

Fig. 5: Test 1 (without advection). Dynamical response of the system. Top: Free-system dynamics (position v/s time). Bottom: Full-order controlled dynamics with reduced-order, minimum time controller.

tering in a considerable way. The controller exhibits a good behavior with respect to output noise (we consider a white noise amplified up to a 30% of the maximum output value), despite the assignment of fast poles for the observer. This is due to the bang-bang structure of the system, where only strong enough output perturbations will cause an estimate to be located on the other side of a given switching surface.

B. Test 2: advection-diffusion-reaction

We perform a second test including advective effects; we set $a_1 = 0.05$, $a_2 = 1.0$, and $a_3 = 0.1$. Note that despite the inclusion of an advective term, such a setting does not deteriorate the stability properties of the original continuous system. A first difference with respect to the previous test can be observed in Figure 7, where it can be seen that the scaling of the decay of the Hankel singular values change, thus suggesting that a larger number of reduced states should be considered for a similar level of accuracy as in the previous example. However, in this test, we will consider three reduced states, as the error associated to the fourth singular value is still within an acceptable range.

The approximation of the associated HJB equation is illustrated in Figure 8, where a cut of the value function along $x_3 = 0$ is shown. The dynamic response of the controlled system is presented in Figures 9 and 10. In Figure 9, it can be seen that although the presence of advection translates into a regulation of the state in finite time, there is a considerable improvement in the performance of the system when a minimum time controller is considered, as both advection and regulation via the control action are combined. In this test case we assume full observation of the state, i.e. our estimation error is uniquely related to the guess of the initial condition $\hat{y}(0)$. Note that, although there is a decay in the estimation error, the effect of advection can be clearly observed in Figure 10. Finally, in a similar way as in the

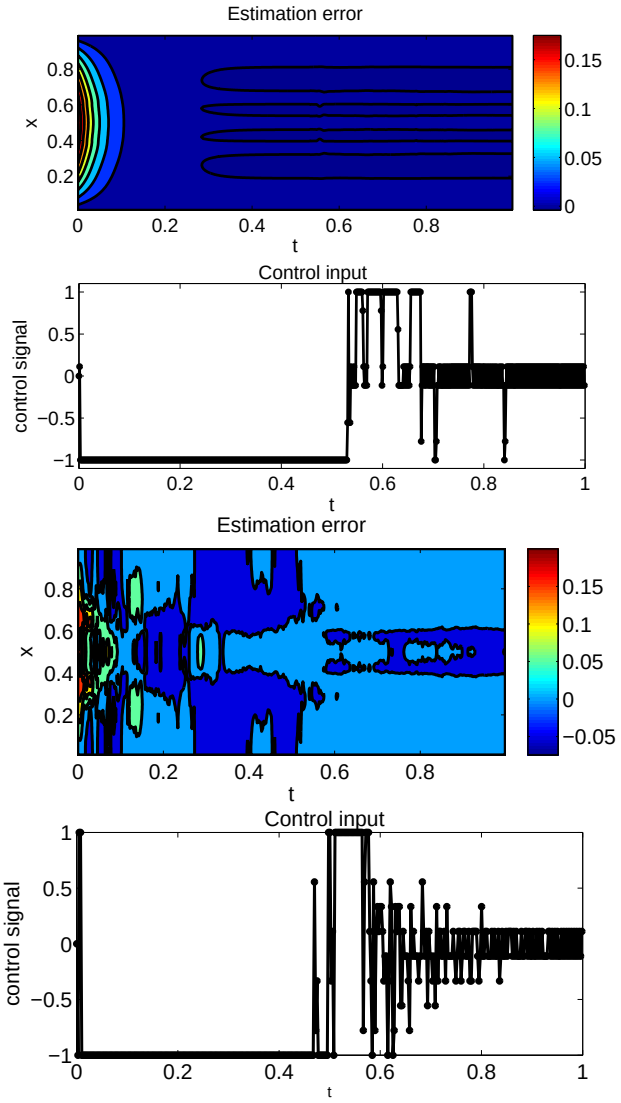

Fig. 6: Test 1 (without advection). Dynamical response of the system. Dynamic discrepancy between internal state of the system and its estimate, and control signal. Tests without noise and with white noise in the output of the plant (top and bottom, respectively).

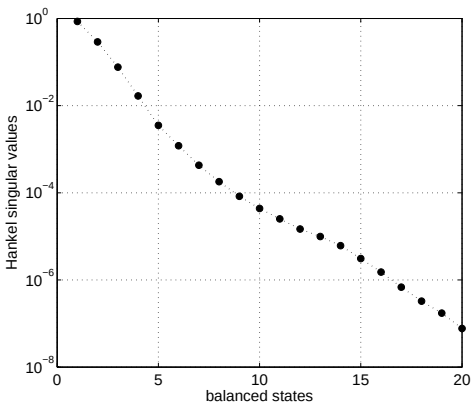

Fig. 7: Test 2 (with advection). Decay of the Hankel singular values in balanced coordinates.

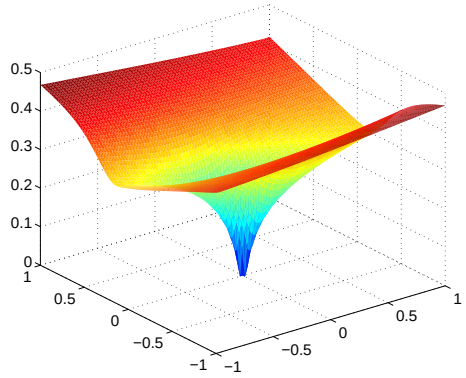

Fig. 8: Test 2 (with advection). Value function: cut at $x_3 = 0$.

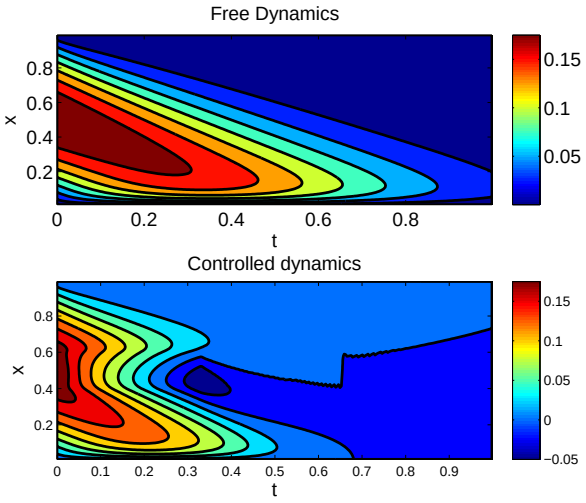

Fig. 9: Test 2 (with advection). Dynamical response of the system. Top: Free-system dynamics (position v/s time). Bottom: Full-order controlled dynamics with reduced-order, minimum time controller.

previous example, Figure 10 shows an approximate bang-bang control signal, with inaccuracies due to the discrete resolution of the HJB equation.

VI. Concluding remarks

We have presented a computationally feasible framework for the implementation of approximate feedback controls in infinite-dimensional systems. Our proposed approach consists in three main steps: semi-discretization and model order reduction of the abstract system dynamics, the application of DPP techniques in order to achieve a characterization of the value function for the lower dimensional optimal control problem, and the implementation of a state observer in order to close the control loop. Every block has a solid computational framework where different numerical methods are available. Our results suggest that the approach is able to yield reasonably robust controllers (in the sense that they can perform well when connected to the large-scale, semi-discrete dynamics), already with a reduced number of states. The proposed approach is flexible as it supports variations in the application of reduction techniques, the optimal feedback

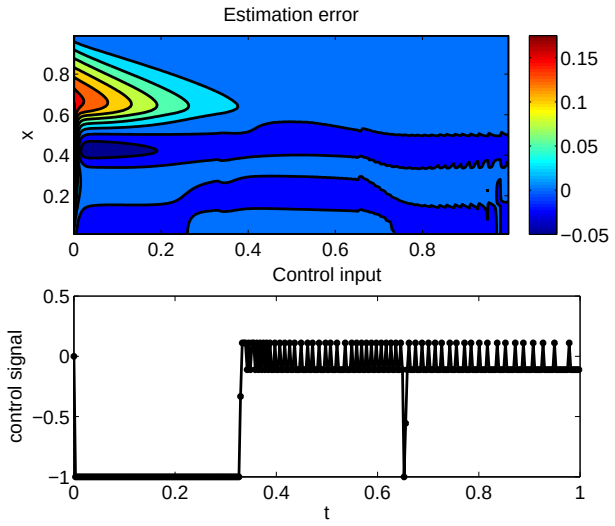

Fig. 10: Test 2 (with advection). Dynamical response of the system. Top: Dynamic discrepancy between internal state of the system and its estimate. Bottom: Control signal.

to be sought, the type of observer considered, and therefore it can be used for a wide class of dynamics and control design problems.

Future extensions of this work shall address the formulation of a precise error quantification connecting the different approximation errors introduced in every step, as also a precise characterization of the robustness properties observed in the numerical essays.

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