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DETERMINATION OF NON-LINEAR NORMAL MODES FOR CONSERVATIVE SYSTEMS

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Abstract. The definition of a non-linear normal mode (NNM) is considered through normal form theory. It is shown that a nonlinear change of variables allows one to span the phase space with the invariant manifolds, and thus to define co-ordinates linked with the NNMs. Invariance property as well as the validity of the asymptotic development used for generating the normal form are illustrated on two examples.

INTRODUCTION

The definition of non-linear normal modes (NNMs) has been subject to various approach since the pioneering work of Rosenberg [1]. The main motivation relies in the fact that, at the non-linear stage, the mode shapes depend on vibration amplitude. This phenomenon, which could be compared with the frequency dependence on motion amplitude, is not made explicit by the usual framework, which consists in projecting the equations of motion onto the linear modes basis.

In this paper, a NNM is defined as an invariant manifold which is tangent to its linear counterpart (the linear eigenspace) at the origin. S. Shaw and C. Pierre used center manifold reduction technique in order to calculate those invariant manifolds [2, 3]. It allows them to compute a single NNM; but the method becomes tedious to handle multiple mode motions [4].

The application of normal form theory allows one to define a non-linear change of variables that cancels non-resonant terms in the non-linear equations of motion. It is then shown that the new defined variables are linked to the invariant manifolds. Hence the geometry of *all* the NNMs, which are computed in a single operation, are contained in this non-linear co-ordinate change. Moreover, attendant dynamics onto the manifolds (*i.e.* the nonlinear modal motions) are simply given by the normal form of the vibratory problem at hand. This procedure greatly simplifies and generalizes the approach proposed in [2, 3]. A similar idea has already been used by Jezequel and Lamarque [5]. A summary of the main results are here provided, as well as two examples. Complete calculations for an N degrees-of-freedom (d.o.f) system are given in [6].

NORMAL FORM THEORY AND NNMs

Normal form theory

A conservative N d.o.f system with quadratic and cubic non-linearities is considered :

$$\forall\, p = 1...N \;\; : \quad \ddot{X}_p + \omega_p^2 X_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N} g_{ij}^p X_i X_j + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N} h_{ijk}^p X_i X_j X_k = 0, \tag{1}$$

It is assumed here that there are no internal resonance relations between the eigenfrequencies $\{\omega_p\}_{p=1...N}$ of the system (1). Normal form theory relies upon Poincaré and Poincaré-Dulac's theorems [7, 8]. The underlying idea is that the physical co-ordinates (X_p, Y_p), where $Y_p = \dot{X}_p$, are not the most appropriate ones to describe the dynamics exhibited by (1). Hence a non-linear change of variables is sought in order to cancel all terms that are not dynamically important. These terms are the non-resonant ones [7, 8].

For an oscillatory problem, one can show that these non-resonant terms are responsible for the loss of invariance of the linear eigenspaces. By invariance we mean that a motion initiated in a subspace always stays in that subspace during its motion. An invariant manifold is then a subset of initial conditions that allows one to define a global motion of a structure with a single displacement-velocity pair. This central property *defines* a NNM [2, 3, 6]. Setting Eqs (1) into its normal form exhibits the co-ordinates linked with the invariant manifolds, which are two-dimensional curved surfaces in the $2N$-dimensional phase space.

The remainder of this article gives a survey of the main analytical results given in [6], where the general equations that define the *normal* co-ordinates (R_p, S_p) are explicited.

Elimination of the quadratic terms

Poincaré's theorem ensures that quadratic terms are non-resonant and thus can be eliminated. This is obtained via the following relations:

$$X_p = U_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N} (a_{ij}^p U_i U_j + b_{ij}^p V_i V_j), \tag{2a}$$

$$Y_p = V_p + \sum_{i=1}^{N}\sum_{j=1}^{N} \gamma_{ij}^p U_i V_j, \tag{2b}$$

For the sake of concision, the expression of the coefficients $\{a_{ij}^p, b_{ij}^p, \gamma_{ij}^p\}_{p=1...N}$ are not given here. They are expressed as functions of the physical parameters $(\omega_p,\;\; g_{ij}^p,\;\; h_{ijk}^p)$ in [6]. One can notice the following general properties: *(i)* Eq. (2) is chosen tangent to the identity, to ensure that the manifolds are tangent to the linear eigenspaces. *(ii)* Expressions of $\{a_{ij}^p, b_{ij}^p, \gamma_{ij}^p\}$ diverge in case of internal resonance, a feature which is completely usual in asymptotic developments. *(iii)* (U_p, V_p) are calculated so as to remain homogeneous to a displacement-velocity pair ($\dot{U}_p = V_p$). They correspond to the co-ordinates linked with a second-order approximation of the invariant manifolds.

Substituting (2) into (1) gives the following dynamics:

$$\dot{U}_p = V_p, \tag{3a}$$

$$\dot{V}_p = -\omega_p^2 U_p - \sum_{i=1}^{N} \sum_{j\geq i}^{N} \sum_{k\geq j}^{N} h_{ijk}^p U_i U_j U_k - \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k\geq j}^{N} \left[A_{ijk}^p U_i U_j U_k + B_{ijk}^p U_i V_j V_k \right]. \tag{3b}$$

The coefficients A_{ijk}^p, B_{ijk}^p are given by:

$$A_{ijk}^p = \sum_{l\geq i}^{N} g_{il}^p a_{jk}^l + \sum_{l\leq i} g_{li}^p a_{jk}^l \quad , \qquad B_{ijk}^p = \sum_{l\geq i}^{N} g_{il}^p b_{jk}^l + \sum_{l\leq i} g_{li}^p b_{jk}^l. \tag{4}$$

As expected, the quadratic terms have been cancelled. Some cubic terms arise from this operation. One can notice that some of them are velocity-dependent, a fact that has already been underlined [2, 3]. The same operation is now carried out, to proceed with the cubic terms.

Processing the cubic terms

A non-linear change of co-ordinates will now be exhibited in order to cancel the cubic non-resonant terms. The specificity of this calculation is that all terms will not be eliminated, due to the particular resonance relations that are always present for systems with eigenspectrum $\{\pm i\omega_p\}_{p\geq 1}$. For example, the order three resonance relation $\omega_1 = (\omega_2 - \omega_2) + \omega_1$ is always fulfilled, leading to an unremovable term $X_2^2 X_1$ in the evolution equation for X_1.

The new displacement/velocity variables are now written (R_p, S_p). They are called the *normal* co-ordinates. The non-linear relationship between (U_p, V_p) and (R_p, S_p) is found to be of the following form:

$$U_p = R_p + \sum_{i=1}^{N} \sum_{j\geq i}^{N} \sum_{k\geq j}^{N} r_{ijk}^p R_i R_j R_k + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k\geq j}^{N} u_{ijk}^p R_i S_j S_k, \tag{5a}$$

$$V_p = S_p + \sum_{i=1}^{N} \sum_{j\geq i}^{N} \sum_{k\geq j}^{N} \mu_{ijk}^p S_i S_j S_k + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k\geq j}^{N} \nu_{ijk}^p S_i R_j R_k. \tag{5b}$$

Assembling (5) together with (2) gives the relationship between the *physical* co-ordinates (X_p, Y_p) and the *normal* co-ordinates (R_p, S_p), which are connected to the order three approximation of the invariant manifolds. Finally, introducing (5) into (3) gives the dynamics which is found to be the normal form of (1). The general equation is not reproduced here for the sake of brevity, and also because the normal form can be written from the knowledge of the linear eigenspestrum only. But interesting features are found within it, and will be treated elsewhere [6].

We show the general result with $N = 2$, as the examples treated below will consider two d.o.f systems:

$$\begin{aligned} \ddot{R}_1 + \omega_1^2 R_1 + &(h_{111}^1 + A_{111}^1) R_1^3 + (h_{122}^1 + A_{122}^1 + A_{212}^1) R_1 R_2^2 \\ &+ B_{111}^1 R_1 S_1^2 + B_{122}^1 R_1 S_2^2 + B_{212}^1 R_2 S_1 S_2 = 0 \end{aligned} \tag{6a}$$

$$\begin{aligned} \ddot{R}_2 + \omega_2^2 R_2 + &(h_{222}^2 + A_{222}^2) R_2^3 + (h_{112}^2 + A_{112}^2 + A_{211}^2) R_2 R_1^2 \\ &+ B_{112}^2 R_1 S_1 S_2 + B_{211}^2 R_2 S_1^2 + B_{222}^2 R_2 S_2^2 = 0 \end{aligned} \tag{6b}$$

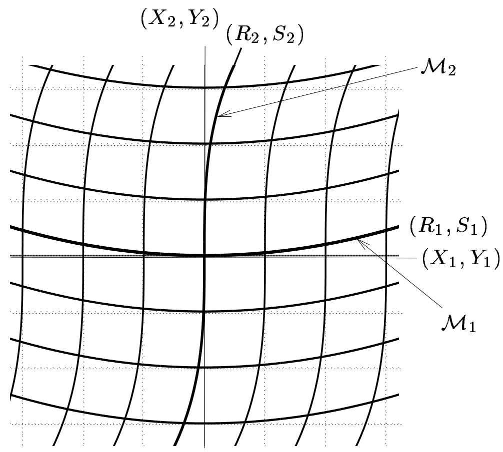

Fig. 1: Sketch of the phase space in the vicinity of the origin. Orthogonal lines represent the linear modal basis. Curved lines tangent at the origin represents the invariant manifolds, i.e. *the NNMs. A curved grid is also represented, in which the normal dynamics will be expressed.*

One can notice that all the terms that are responsible for the loss of invariance have been cancelled, as well as the presence of velocity-dependent terms in the normal dynamics. Single non-linear modal dynamics is now simply given by setting all others co-ordinates to zero, as in a linear case. For example, the first NNM is found by setting $R_2 = S_2 = 0$. This gives the dynamics onto the first manifold (by substituting into (6a)), as well as its geometry in phase space:

$$X_2 = a_{11}^2 R_1^2 + b_{11}^2 S_1^2 + r_{111}^2 R_1^3 + u_{111}^2 R_1 S_1^2 \quad (7a)$$
$$Y_2 = \gamma_{11}^2 R_1 S_1 + \mu_{111}^2 S_1^3 + \nu_{111}^2 S_1 R_1^2 \quad (7b)$$

The situation is sketched in figure 1. The linear orthogonal eigenspaces, graduated by the physical co-ordinates (X_p, Y_p) are represented by lines for convenience, but they are in fact planes. The invariant manifolds (denoted $\mathcal{M}_1$ and $\mathcal{M}_2$) are represented by the curved lines. Non-linear single-mode orbits are contained within these invariant manifolds.

Throughout the section, it has been shown that normal form theory gives: *(i)* the geometry of all invariant manifolds, *(ii)* co-ordinates linked with them and thus the bending of the whole phase space and *(iii)* the reduced dynamics with the new co-ordinates. These calculations have been found to be consistent with previous approaches developped by other authors [2, 4, 9]. It will now be applied on examples.

EXAMPLES

System with cubic non-linearity

The first example considers a two d.o.f system with cubic non-linearity, shown in figure 2. The equations of motion are given by:

$$\dot{X}_1 = Y_1 \quad , \quad \dot{Y}_1 = -(1+k)X_1 + kX_2 - gX_1^3 \quad (8a)$$
$$\dot{X}_2 = Y_2 \quad , \quad \dot{Y}_2 = kX_1 - (1+k)X_2 \quad (8b)$$

The first thing to do is a linear decoupling, that is to express Eq. (8) into the linear modal basis,

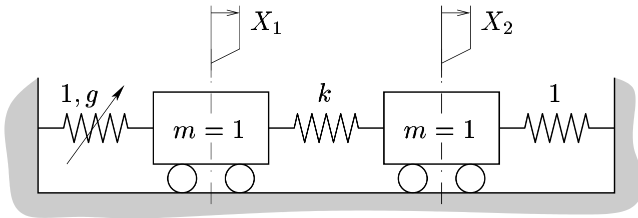

Fig. 2: Physical representation of example 1.

defined by the two following linear modal motions : $X_1 = U_1 + U_2$, $X_2 = U_1 - U_2$. This gives the following system:

$$\dot{U}_1 = V_1 \quad , \quad \dot{V}_1 = -U_1 - \frac{g}{2}(U_1 + U_2)^3 \tag{9a}$$

$$\dot{U}_2 = V_2 \quad , \quad \dot{V}_2 = -(1+2k)U_2 - \frac{g}{2}(U_1 + U_2)^3 \tag{9b}$$

The linear frequencies are $\omega_1 = 1$ and $\omega_2 = \sqrt{1+2k}$. The assumption of no internal resonance is fulfilled as long as $k \neq 0$, $k \neq 4$ and $k \neq -4/9$. The results of the precedent sections give the non-linear relation that permits to describe the bending of the whole phase space:

$$U_1 = R_1 + \frac{(3+7k)g}{4k(9k+4)}R_2^3 - \frac{3(1-k)g}{4k(k-4)}R_1^2 R_2 + \frac{3g}{2k(k-4)}R_1 S_1 S_2 - \frac{9g}{4k(k-4)}S_1^2 R_2 + \frac{3g}{4k(9k+4)}R_2 S_2^2 \tag{10a}$$

$$V_1 = S_1 + \frac{3g}{4k(9k+4)}S_2^3 + \frac{3(k-3)g}{4k(k-4)}R_1^2 S_2 + \frac{3(1-k)g}{2k(k-4)}R_1 R_2 S_1 - \frac{3g}{4k(k-4)}S_1^2 S_2 + \frac{3(1+3k)g}{4k(9k+4)}R_2^2 S_2 \tag{10b}$$

$$U_2 = R_2 + \frac{(3-k)g}{4k(k-4)}R_1^3 + \frac{3g}{4k(k-4)}R_1 S_1^2 - \frac{3(1+3k)g}{4k(9k+4)}R_1 R_2^2 - \frac{9g}{4k(9k+4)}R_1 S_2^2 + \frac{3g}{2k(9k+4)}S_1 R_2 S_2 \tag{10c}$$

$$V_2 = S_2 + \frac{3g}{4k(k-4)}S_1^3 + \frac{3(1-k)g}{4k(k-4)}R_1^2 S_1 + \frac{3(1+3k)g}{2k(9k+4)}R_1 R_2 S_2 - \frac{3(3+7k)g}{4k(9k+4)}S_1 R_2^2 - \frac{3g}{4k(9k+4)}S_1 S_2^2 \tag{10d}$$

Applying this co-ordinate change gives the normal dynamics, up to order 3:

$$\dot{R}_1 = S_1 \quad , \quad \dot{S}_1 = -R_1 - \frac{g}{2}R_1^3 - \frac{3g}{2}R_2^2 R_1 \tag{11a}$$

$$\dot{R}_2 = S_2 \quad , \quad \dot{S}_2 = -(1+2k)R_2 - \frac{g}{2}R_2^3 - \frac{3g}{2}R_1^2 R_2 \tag{11b}$$

Figure 3 shows the results of simulations concerning the invariance property, with $g = 1/3$ and $k = 1$. The first column is related to the linear modal subspace. Inital condition has been taken along the first linear mode, namely : $U_1 = 1$ and all other co-ordinates to zero. Figure 3(a) shows the projection of the motion, calculated in *physical* co-ordinates (*i.e.* with (8)), onto the first

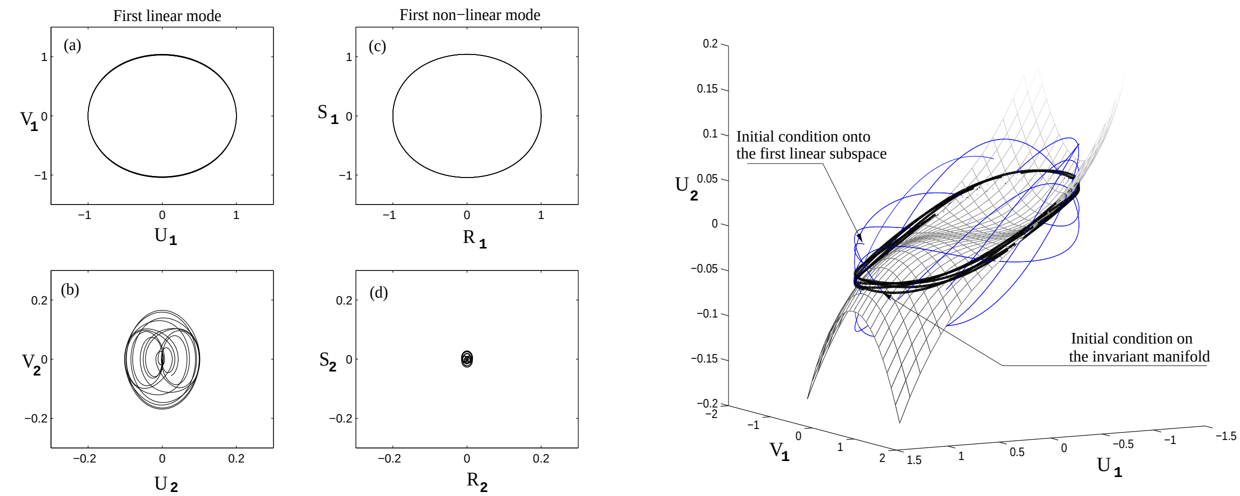

Fig. 3: (a)-(d): projections onto linear and non-linear subspaces of a motion initiated along the first linear mode(a)-(b) ; and along the first non-linear mode (c)-(d). Representation of the two precedent simulations in (U_1, V_1, U_2) space, showing the invariance property for the first NNM.

linear subspace (U_1, V_1). Figure 3(b) shows the residual contribution on the second linear subspace (U_2, V_2), which is clearly not negligible. This is the effect of non-resonant terms that have been cancelled through the nonlinear change of co-ordinates.

The same simulation is now conducted with the first non-linear normal mode. An initial position along the first non-linear manifold is given by : $R_1 = 1$, all the other co-ordinates set to zero. Equations (10) allows one to calculate the corresponding initial condition in *physical* co-ordinates. For the first non-linear mode, this gives : $X_1 = 0.944$, $X_2 = 1.055$, $Y_1 = Y_2 = 0$. The dynamics is integrated numerically with (8) and then projected back onto the non-linear invariant manifolds, with the inverse relation of (10). Figure 3(c) shows the motion along the first non-linear mode, and Figure 3(d) the residual contribution along the second non-linear mode. Comparing (b) and (d) shows that a factor 7 has been gained for the invariance. The fact that the motion in (d) is not completely zero is due to the fact that only a third-order approximation of the invariant manifold is computed. The results of these two simulations are plotted in the (U_1, V_1, U_2) space. The order-three approximation of the first invariant manifold is represented by a surface. The two calculated trajectories are also shown. One can see that taking the initial condition on the first linear mode results in a whirling trajectory. The motion started on the invariant manifold, the first NNM, stay on it for all time.

A system with quadratic and cubic non-linearities

The two d.o.f system considered here is composed of a mass m connected to a frame by two geometrically non-linear springs (Fig. 4(a)). The potential energy of spring i $(i \in \{1, 2\})$ is assumed to be:

$$\mathcal{E}_p = \frac{1}{2} k_i \, l_0^2 \, e^2, \quad \text{with} \quad e = \frac{1}{2} \frac{l_i^2 - l_0^2}{l_0^2}. \tag{12}$$

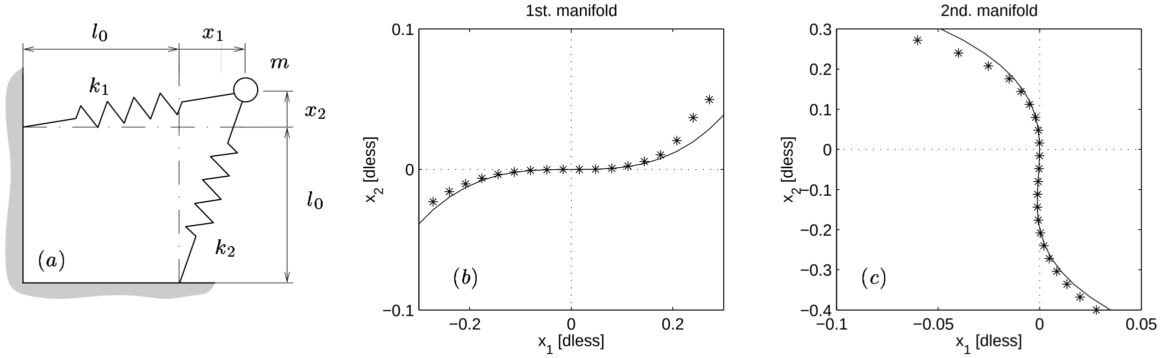

Fig. 4: (a) - The two d.o.f mass/spring system. (b) - Third order approximation (solid lines) and true position ('') of the first invariant manifold. (c) -* idem *for the second invariant manifold.*

k_i is the linear stiffness of spring i, and l_i is its deformed length. The tension in the springs is then:

$$N_i = -\frac{\partial \mathcal{E}_p}{\partial l_i} = -\frac{1}{2} k_i \, l_0 \left[\left(\frac{l_i}{l_0} \right)^3 - \frac{l_i}{l_0} \right] = -k_i \left[\Delta l_i + \frac{3}{2} \frac{\Delta l_i^2}{l_0} + \frac{1}{2} \frac{\Delta l_i^3}{l_0^2} \right], \quad \text{with} \quad \Delta l_i = l_i - l_0. \tag{13}$$

Applying Lagrange's equations to the mass,

$$\mathcal{L}_i : \quad \left(\frac{d}{dt} \frac{\partial}{\partial \dot{x}_i} - \frac{\partial}{\partial x_i} \right) (\mathcal{E}_k + \mathcal{E}_p) = 0, \tag{14}$$

with $\mathcal{E}_k$ denoting the kinetic energy of mass m, one obtains the following system, with quadratic and cubic non-linearities:

$$\ddot{X}_1 + \omega_1^2 X_1 + \frac{\omega_1^2}{2}(3X_1^2 + X_2^2) + \omega_2^2 X_1 X_2 + \frac{\omega_1^2 + \omega_2^2}{2} X_1 (X_1^2 + X_2^2) = 0 \tag{15a}$$

$$\ddot{X}_2 + \omega_2^2 X_2 + \frac{\omega_2^2}{2}(3X_2^2 + X_1^2) + \omega_1^2 X_1 X_2 + \frac{\omega_1^2 + \omega_2^2}{2} X_2 (X_1^2 + X_2^2) = 0 \tag{15b}$$

$X_1 = x_1/l_0$ and $X_2 = x_2/l_0$ are dimensionless displacements defining the position of the mass, $w_1^2 = k_1/m$ and $w_2^2 = k_2/m$ are the natural frequencies of the system. The system is naturally linearly uncoupled, as the two linear modes of motion are X_1 and X_2. It is a consequence of the orthogonal configuration of the two springs at rest.

Applying the results of the first section leads to a third-order approximation of the invariant manifold. Similar numerical simulations than the ones of Fig. 3 can be obtained in the case of the present system. However, this third-order approximation of the manifolds fails when the non-linearity is raised. This problematic feature has also been observed in the first example. Here, we show quantitatively in Fig. 4(b, c) the difference between the third-order approximation and the position of the true invariant manifold in phase space.

Solid lines represent the third-order approximation, and the stars ('*') corresponds to the position of the true invariant manifold. These points have been found numerically by systematically investigating different initial conditions in phase space. The retained points are those, *e.g.* for the first non-linear mode, which give the minimum contribution onto the second non-linear mode. One can notice that the third-order approximation sometimes quickly diverge. This can be remedied by using a non-linear Galerkin method to fit the NNM. A recent study of Peshek *et al.* gives a promising account for this, which really exploits the full potential of the invariant manifold approach [10].

CONCLUSION

Nonlinear normal modes have been defined with the help of normal form theory, for conservative systems. It generalizes the asymptotic approach proposed in [2, 3], since all NNMs are computed in a single operation. Moreover, it has been shown that the dynamics, for an assembly of N non-linear oscillators, is simply given by the normal form of the problem considered. This point is important since the normal form depends on the eigenspectrum of the linear evolution operator only. Thus, *ex-nihilo* models can be simply exhibited. The main drawback of the method, which relies upon an asymptotic development, has been underlined through the examples.

However, the full potential of describing the dynamics within a curved reference system spanned by the invariant manifolds will be expressed when considering reduced-order models of systems having large number of d.o.f. Multiple modes motions are effectively easy to define and to compute. Moreover, the case of internal resonance does not lead to tedious calculations and are also easily taken into account. The results presented here are important in this sense, since normal form is the cornerstone of very general results.

Acknowledgment: The authors would like to thank Bruno Cochelin from LMA-Marseille for suggesting the second example.

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