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Abstract

Large-amplitude vibrations of a clamped-free beam are considered. Reduced-order models (ROMs) are derived for this problem, within the framework of non-linear normal modes (NNMs), defined as invariant manifolds in phase space. The method of real normal form theory, which allows computation of all NNMs in a single operation, is used [1, 2]. A specific development enables to handle the non-linear inertia terms stemming from the large rotation beam model. The dynamics onto the manifold is derived up to order five. Non-linear mode shapes are exhibited, as well as frequency-amplitude relationships. Finally, the case of a harmonic base-excitation is considered.

GOVERNING EQUATIONS

In non-dimensional form, the equation governing planar motion of a cantilever beam is given by [3]:

$$\ddot{w} + w'''' = -\left[w'w''^2 + w'''w'^2\right]' - \frac{1}{2}\left[w'\int_1^x \frac{\partial^2}{\partial t^2}\left(\int_0^u w'^2(y)\mathrm{d}y\right)\mathrm{d}u\right]', \quad (1)$$

where $w(x,t)$ is the transverse displacement, $\dot{w}$ denotes derivation with respect to time t and w' derivation with respect to space x. The dimensionless variables have been choosen such that a displacement w equal to 1 in the model refers to a real displacement equal to the length of the beam. Associated boundary conditions are:

$$\forall\, t,\; w(0,t) = w'(0,t) = w''(1,t) = w'''(1,t) = 0. \quad (2)$$

The aim of the present paper is to exhibit reduced-order models of this equation by considering non-linear normal modes as an invariant-based span of the phase space. Real normal form theory allows derivation of an asymptotic approach to the complete non-linear change of coordinates, according to [1, 2, 4]. Hence, the calculations presented herein are a generalization of earlier results given in [5, 6]. It is also a generalization of the method presented in [1, 2] which were designed for geometric polynomial non-linearities in displacement, since specific developments enables here to handle the non-linear inertia terms.

The first step of the computation is to set apart the spatial dependence by projection of Eq. (1) onto the complete set of eigenfunctions defined by the linear part. The displacement w is expanded as:

$$w(x,t) = \sum_{p=1}^{+\infty} X_p(t)\Phi_p(x)\,, \quad (3)$$

where the linear eigenmodes Φ_p are not recalled for the sake of brevity (see *e.g.* [3, 5, 6]). After projection and truncation to N linear modes (where N is assumed to be large), the problem writes, $\forall\; p = 1...N$:

$$\ddot{X}_p + \omega_p^2 X_p = -\sum_{i,j,k=1}^{N} h_{ijk}^p X_i X_j X_k - \sum_{i,j,k=1}^{N} f_{ijk}^p (X_i X_j \ddot{X}_k + X_i \dot{X}_j \dot{X}_k). \quad (4)$$

The coefficients of the non-linear terms are:

$$h_{ijk}^p = \int_0^1 (\Phi_i' \Phi_j'' \Phi_k'' + \Phi_i''' \Phi_j' \Phi_k')' \Phi_p \mathrm{d}x, \quad (5)$$

$$f_{ijk}^p = \int_0^1 \left[\Phi_i' \int_1^x \int_0^u \Phi_j'(y)\Phi_k'(y)\mathrm{d}y\mathrm{d}u\right]' \Phi_p \mathrm{d}x. \quad (6)$$

They are numerically computed for N=15, which will be the number of linear modes retained in the following.

NON-LINEAR NORMAL MODES

Inertial non-linearity

Real normal form theory, as defined in [1, 2], allows computation of a complete change of coordinates, from the phase space into itself. After this operation, the dynamics is expressed in a curved invariant-based span, where each of the N subspace (say k) is the k^{th} invariant manifold. Within this framework, truncation of EDOs governing the dynamics can be properly realized. Retaining only one subspace (k) recovers the equation of the manifold which defines the k^{th} NNM, as well as the dynamics onto it.

In order to use developments presented earlier in [1, 2], which were defined for geometric non-linearities only, Eq. (4) has to be put under the usual form $\dot{\boldsymbol{\xi}} = \mathbf{F}(\boldsymbol{\xi})$ which defines dynamical systems. This is realized by writing Eq. (4) as:

$$(\mathbf{I}_N + \boldsymbol{\Xi}_N(\mathbf{X}))\ddot{\mathbf{X}} = \mathbf{G}(\mathbf{X})\,, \quad (7)$$

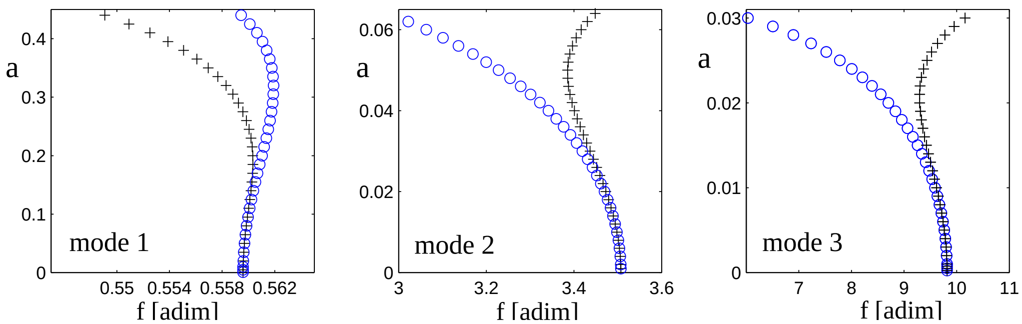

Figure 1: Backbone curves for the first three modes. $\circ$: order-three dynamics, $+$: order-five dynamics.

where $\mathbf{X} = (X_1, ..., X_N)^t$, $\mathbf{I}_N$ is the $N \times N$ identity matrix, $\Xi_N(\mathbf{X})$ is a $N \times N$ matrix whose element (i, j) reads (Einstein's notation is used when necessary): $f^i_{uvj} X_u X_v$, and $\mathbf{G}(\mathbf{X})$ is a vector whose p^{th} element reads:

$$\mathbf{G}_p(\mathbf{X}) = -\omega_p^2 X_p - h^p_{ijk} X_i X_j X_k - f^p_{ijk} X_i \dot{X}_j \dot{X}_k. \quad (8)$$

Assuming that $\mathbf{X}$ is small, Eq. (7) is formally written, up to order five:

$$\ddot{\mathbf{X}} = (\mathbf{I}_N - \Xi_N(\mathbf{X}) + \Xi_N(\mathbf{X}).\Xi_N(\mathbf{X}))\mathbf{G}(\mathbf{X}), \quad (9)$$

where the expanded developments are not reproduced for the sake of brevity. Within this formulation, the computation of the NNMs, as proposed in [1, 2], is now possible.

Order-five dynamics

The computation of the NNMs through real normal form theory is carried out by an order-three asymptotic development [1, 2], formally written:

$$\begin{pmatrix} X_p \\ Y_p \end{pmatrix} = \begin{pmatrix} R_p \\ S_p \end{pmatrix} + \begin{pmatrix} P_p^{(3)}(R_i, S_i) \\ Q_p^{(3)}(R_i, S_i) \end{pmatrix} \quad (10)$$

(R_p, S_p) are the new *normal* coordinates, related to the p^{th} NNM, and $P_p^{(3)}$ and $Q_p^{(3)}$ are polynomials whose expression can be found in [1, 2]. Substituting Eq. (10) into (9) allows expression of the dynamics with the new generalized *normal* displacement-velocity variables (R_p, S_p). Efficient truncations have to be realized on this last system of EDOs [2, 4].

As already mentioned in [4], for problems with odd non-linearities only (as is the dynamics defined by Eq. (9)), although the non-linear change of coordinates is expressed up to order three, the dynamics onto *a single* manifold can be found up to order five without invoking a great amount of algebra. For the cantilever beam problem, the dynamics of the p^{th} NNM up to order five reads:

$$\ddot{R}_m + \omega_m^2 R_m + (h^m_{mmm} - \omega_m^2 f^m_{mmm}) R_m^3 + f^m_{mmm} R_m S_m^2 + \Theta_m R_m^5 + \Upsilon_m R_m^3 S_m^2 + \Gamma_m R_m S_m^4 = 0. \quad (11)$$

The coefficients introduced are equal to:

$$\Theta_m = (\tilde{h}^m_{kmm} + \tilde{h}^m_{mkm} + \tilde{h}^m_{mmk}) r^k_{mmm} - f^m_{mmk} \tilde{h}^k_{mmm}, \quad (12)$$

$$\Upsilon_m = (\tilde{h}^m_{kmm} + \tilde{h}^m_{mkm} + \tilde{h}^m_{mmk}) u^k_{mmm} - f^m_{mmk} f^k_{mmm} + f^m_{kmm} r^k_{mmm} + f^m_{mkm} \nu^k_{mmm} + f^m_{mmk} \nu^k_{mmm}, \quad (13)$$

$$\Gamma_m = f^m_{kmm} u^k_{mmm} + f^m_{mkm} \mu^k_{mmm} + f^m_{mmk} \mu^k_{mmm}, \quad (14)$$

where $\{r^p_{ijk}, u^p_{ijk}, \mu^p_{ijk}, \nu^p_{ijk}\}$ are the coefficients of $P_p^{(3)}$ and $Q_p^{(3)}$, and $\tilde{h}^p_{ijk} = h^p_{ijk} - \omega_k^2 f^p_{ijk}$.

Backbone curves for the first three modes are shown in Figure 1. It should be noted that the order-three dynamics (*i.e.* Eq. (11) truncated to order three) is exactly the same as the dynamics obtained by retaining a single linear mode. It is observed that the first mode displays a hardening behaviour, whereas the others display softening behaviour, a classical result for the cantilever beam (see *e.g.* [7]). The correction bringed by the NNM formulation on the backbone curves appears when considering the order-five dynamics. The curvature changes of the frequency-response relationships have to be ascertained by a stability analysis of the invariant manifolds by using Floquet theory for example. This task is beyond the scope of the present paper.

Non-linear mode shapes

The main advantage of the method of real normal form is that the complete change of coordinates defined by Eq. (10) is computed in a single operation. Thus the effect of possible internal resonances are easily taken into account by keeping the NNMs involved in a multi-dof invariant manifold. The normal dynamics is readily given by keeping the resonant terms [1, 2]. For the cantilever beam problem, order-three internal resonances are found in the eigenspectrum recalled in Table 1. However they don't give rise to invariant-breaking terms. The only internal resonance susceptible to create an invariant-breaking term is $\omega_4 = 2\omega_3 + 2.49$, which has not been considered here. Finally, single NNM motions are possible and are now briefly studied.

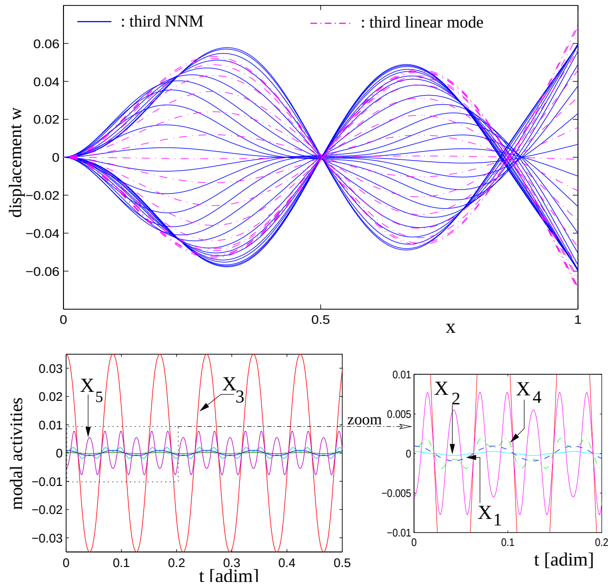

Figure 2: Beam displacement during a half-period of free oscillation initiated on the third NNM, and time histories of the associated modal amplitudes

Table 1: Six first eigenfrequencies.

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
3.51	22.03	61.69	120.9	199.85	298.55

When considering a single NNM, the results reduce to those presented in earlier studies [5, 6]. Figure 2 shows how the beam vibrates through a half-period of motion along the third NNM. This figure is obtained by simulating Eq. (11) with $m = 3$, then Eqs. (10) and (3) are applied to recover the displacement $w(x, t)$. Modal activities are thus directly available and are also represented. One can notice that the beam's shape varies continuously with time, and that the coupling with the fifth linear mode is proeminent.

HARMONIC BASE-EXCITATION

The aim of this section is to develop a complete model for the cantilever beam subjected to a harmonic base-excitation. Reduced-order models (ROMs) are considered by using the precedently-defined NNMs. These ROMs are computed in order to compare model predictions with a series of measurements realized by Pai and Lee [8], as well as to give insight into some open questions raised by their study.

The harmonic base-excitation $w_0(t) = Q\cos(\Omega t)$ give rise to inhomogeneous boundary conditions. One defines the relative displacement $\tilde{w}(x, t)$ of the beam with respect to the base as: $w(x, t) = \tilde{w}(x, t) + w_0(t)$. The problem is then solved for $\tilde{w}(x, t)$, which can be expanded onto the linear modes basis: $\tilde{w}(x, t) = \sum X_p(t)\Phi_p(x)$. It reads, $\forall\, p = 1..N$:

$$\ddot{X}_p + \omega_p^2 X_p + h^p_{ijk} X_i X_j X_k + f^p_{ijk}(X_i X_j \ddot{X}_k + X_i \dot{X}_j \dot{X}_k) = Q_p(t), \quad (15)$$

where $Q_p(t) = -\ddot{w}_0 \int_0^1 \Phi_p(x)\mathrm{d}x$.

In order to use the results of the previous section, an inversion procedure, similar to that realized through Eqs (7)-(9), has to be tackled. As a forcing term is now present in the right-hand side of the dynamics, only third-order developments are now considered.

Damping has to be added for completing the model. It has been choosen here to define a *non-linear modal damping* as the rate of decay of the trajectories *onto* each invariant manifold. Hence a viscous damping term, of the form $\mu_p \dot{R}_p$, is added to the model, *after* the non-linear change of coordinates defined by Eq. (10). In the simulations, coefficients μ_p have been taken equal to 0.2.

ROMs are now selected by keeping an arbitrary number of non-linear modes for simulating the dynamics. A usual procedure when dealing with forced oscillations is to keep a single forced oscillator to describe the dynamics. Keeping a single linear mode leads to erroneous results mainly because the linear eigenspaces are not invariant in the non-linear range. NNMs have been introduced in order to prevent from those mistakes. It allows, in particular, prediction of the correct trend of non-linearity, while keeping a single oscillator [2]. Unfortunately, keeping a single NNM in the forced case –even if it overcomes the precedent mistakes– is also a too severe truncation. This is because the forcing term, which is naturally present on each non-linear oscillator, introduces energy on each normal coordinate. These contributions, although small (and often neglected under the argument that the forcing frequency is far from natural frequency), should not be cancelled.

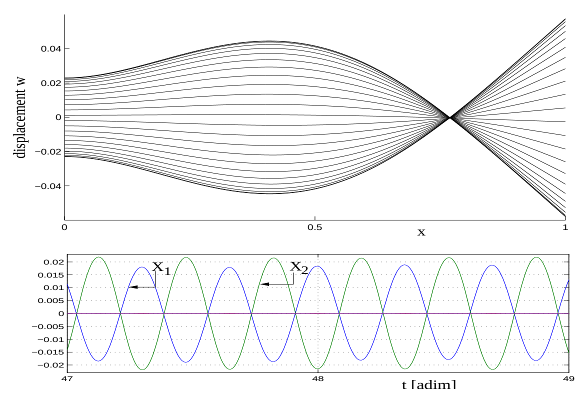

Figure 3: Beam motion for $\Omega = 18 < \omega_2$, and $Q = 0.023$; and modal activities X_i, for i=1...5

The selected ROM is composed of two NNMs. We first study the motion in the vicinity of the second eigenfrequency, see figure 3 for $\Omega < \omega_2$, and figure 4 for $\Omega > \omega_2$. The simulations, for which the first two NNMs have been retained (R_1 and R_2), display very good agreement with the observations reported by Pai and Lee (see fig. 7 and 8 in [8] for the similar cases): the motion mainly consists of the two first linear modal coordinates X_1 and X_2 (all the others are negligible compared to them), and they are out-of-phase before the resonance, and in-phase after the resonance. When $\Omega \simeq \omega_2$, a slight phase difference between X_1 and X_2 is observed in our simulations and in the experiments.

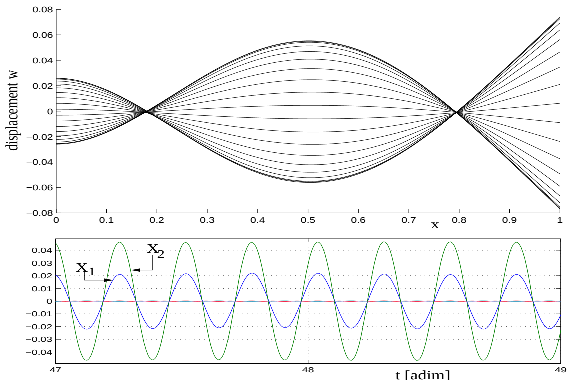

Figure 4: Beam motion for $\Omega = 24 > \omega_2$, and $Q = 0.026$; and modal activities X_i, for i=1...5

Figure 5 shows the simulated beam motion in the vicinity of the third eigenfrequency: $\Omega = 57$. The selected NNMs for the ROM are the third and the fourth: R_3 and R_4. For comparison, the beam vibratory motion obtained by retaining the third and fourth linear modes (X_3 and X_4) is also shown. Significative differences are observed. The magnitude of the modal acitivity of X_1 should also be noticed, and can be related to the experimental observations of Pai and Lee. All these results point out that a ROM based on NNMs is able to predict the main features of the vibration.

CONCLUSION

This paper examines the NNM formulation for a cantilever beam as well as application of this methodology for computing ROMs in the case of a harmonic base-excitation. For free undamped vibration, the concept of NNMs is now well established. It has been shown here that the non-linear inertia terms can be handled within the framework of real normal form theory. However, asymptotic developments act as a brake upon generalization of this procedure, due to its limited applicability range [4]. The problem of external forcing has been briefly considered, in relation with experimental observation, and in a manner which differs from recent results presented in [9], where time-dependent invariant manifolds are computed. It has been underlined that residual forcing terms, which are always present except in the unrealistic case of a forcing whose spatial dependence is defined by an eigenmode shape function, lead to residual contributions on non-directly excited NNMs that cannot be neglected. This implies in particular that the work realized to recover subspace's invariance (expressed into Eq. (10)), loses its powerful meaning, and that NNM mode shapes (such as represented on figure 2) are not observable in forced oscillations. However, ROMs are always computable, and give nonetheless results that are in good agreement with experiments.

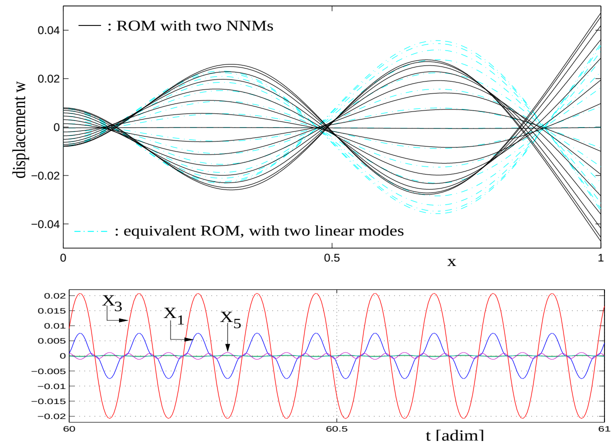

Figure 5: Top: beam motion for $\Omega = 57$ and $Q = 0.008$, model with two NNMs (R_3 and R_4), and model with two linear modes X_3 and X_4. Bottom: modal activities corresponding to the motion computed with two NNMs.

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