An upper bound for normal form asymptotic expansions

Claude-Henri Lamarque, Cyril Touzé, Olivier Thomas

11$^{\text{TH}}$ CONFERENCE
on
DYNAMICAL SYSTEMS THEORY AND APPLICATIONS
December 5-8, 2011. Łódź, Poland

An upper bound for normal form asymptotic expansions

Claude-Henri Lamarque, Cyril Touzé, Olivier Thomas

Abstract: In the nineties, analytical methods based on normal form theory have been used to obtain approximate expressions of free or forced responses of nonlinear (smooth) mechanical systems. A large amount of interesting results can be obtained by these approaches, however no practical limits of validity are available. In this paper, a method to obtain simple upper bounds for amplitudes of changes of coordinates is proposed. Simple conservative mechanical examples are provided to illustrate the bounds estimation and their validity.

1. Introduction

Nonlinear analysis of smooth dynamical systems with finite number of degrees of freedom based on normal form approaches has been developped since the nineties [1,2]. These methods lead to model reduction, identification and easier understanding of nonlinear phenomena in vibrations of mechanical systems [3]. As it is for most of analytical methods, no practical limits of validity is available for approaches based on normal forms. This constitutes the only drawback of these methods since the range of amplitude validity has to be checked versus direct numerical integration of the original system. This paper intends to provide an upper bound for validity limits of approaches based on normal form theory.

2. Classical normal form theory

In this section we recall basic normal form theory for a finite dimensional nonlinear dynamical system written as :

$$\frac{dx}{dt} = f(x) = L_0 x + g(x) \tag{1}$$

where $x \in \mathcal{E}$, $\mathcal{E} = \mathbb{R}^n$ or $\mathbb{C}^n$ is the phase space, L_0 denotes $Df(0)$ the jacobian matrix of f in $x = 0$ and g stands for the higher-order terms (at least order 2).

We assume that f is given by power series expansion (finite sum or analytical function). This frame permits us to deal with usual cases where L_0 is a real matrix or a complex matrix (e.g. in diagonal form).

2.1. Normal form for reduced equations

Let us deal with simplification of the equation (1) by normal form theory. The principle of normal form is as follows: let us give an order k. Let us determine polynomials Φ and R of degree $\leq k$ such as

$$\Phi, P: \quad \mathcal{E} \longrightarrow \mathcal{E} \tag{2a}$$

$$\text{with}: \quad \Phi(0) = R(0) = 0, \quad D\Phi(0) = DR(0) = 0, \tag{2b}$$

and R is as simple as possible (if possible 0) and so that change of variables $x = u + \Phi(u)$ transforms (1) into

$$\frac{du}{dt} = L_0 u + R(u) + o(\mid u \mid^k). \tag{3}$$

Let us replace in equation (1):

$$\frac{d}{dt}(u + \Phi(u)) = (I + D\Phi).\frac{du}{dt} = f(u + \Phi(u)). \tag{4}$$

Then since final result is given by (3), we obtain

$$(I + D\Phi).(L_0 u + R(u)) = f(u + \Phi(u)). \tag{5}$$

where I denotes identity operator. In practice this equality is solved degree by degree via Taylor expansions (see e.g. Iooss and Adelmeyer [1]) leading to linear systems. The associate Fredholm alternative provides both the resonant terms R and the normal transforms Φ. The latter one is not uniquely defined due to Fredholm alternative and the typical structure of the problem. For non-resonant terms that can be cancelled through the change of coordinates, a unique solution is at hand; whereas for resonant terms, choice for Φ is not unique. The usual and simplest choice is given by $\Phi = 0$, leading to the simplest expression of normal transform.

For practical use of such calculations, one should have results about convergence of the process. Mathematical questions of convergence can be found in Brjuno's works (e.g. [4]). These kind of results are beyond practical use. Here we intend to give a practical upper bound estimated value for normal coordinate u.

2.2. Practical convergence

Normal form calculations are made in practice up to a given finite degree k. So writing $R = R_2 + \cdots + R_k + \ldots$, and $\Phi = \phi_2 + \cdots + \phi_k + \ldots$, all the R_j and ϕ_j $(2 \leq j \leq k)$ are computed degree by degree . Let us note that in fact equation (4) has to be understood as

$$\frac{du}{dt} = (I + D\phi(u))^{-1}(f(u + \Phi(u))) \tag{6}$$

leading to $\frac{du}{dt} = L_0 u + R(u)$. So we propose to set a boundary for practical use of normal transform, associated to the distance of u from $0 \in \mathcal{E}$ so that $I + D\phi(u)$ becomes singular and $(I + D\phi(u))^{-1}(f(u + \Phi(u)))$ keeps also a singular point (it means the limit of the term $f(u+\Phi(u))$ when u tends to singular value of $(I+D\phi(u))$ does not suppress the singularity).

Since in approximated problems we compute only terms $\phi_2, \ldots, \phi_k$ and since we collect nonlinearities according to increasing degrees in $f(u + \Phi(u))$ we propose in pratice to look for u so that

$$\Delta(u) = \det(I + D\phi_2(u) + \cdots + D\phi_k(u)) = 0 \tag{7}$$

and so that

$$\lim_{v \longrightarrow u, \Delta(u)=0} \left| \frac{T(f(v + \phi_2(v) + \cdots + \phi_k(v)))}{\Delta(v)} \right| = +\infty, \tag{8}$$

where $T(f(v+\phi_2(v)+\cdots+\phi_k(v)))$ denotes the truncated Taylor expansion of $f(v+\phi_2(v)+\cdots+\phi_k(v))$ up to order k.

3. First example: the Duffing equation

Let us consider a single degree-of-freedom (dof) mechanical system with cubic nonlinearity:

$$\frac{d^2 w_1}{dt^2} + \omega_1^2 w_1 + c w_1^3 = 0. \tag{9}$$

Normal form calculation up to order 3 are based on the following changes of variables:

$$\dot{w}_1 = \lambda_1 x_1 + \lambda_2 x_2, w_1 = x_1 + x_2, \lambda_1 = \overline{\lambda_2} = -i\omega_1, i^2 = -1. \tag{10}$$

$$x_1 = u_1 + \phi_1(u_1, u_2), x_2 = u_2 + \phi_2(u_1, u_2), \tag{11}$$

where

$$\phi_1(u_1, u_2) = -1/8\,\frac{c{u_2}^3}{{\omega_1}^2} - 3/4\,\frac{c u_1 {u_2}^2}{{\omega_1}^2} + 1/4\,\frac{c{u_1}^3}{{\omega_1}^2}, \phi_2 \text{ conjugate of } \phi_1. \tag{12}$$

Normal form up to order 3 is given by:

$$\frac{du_1}{dt} = -i\omega_1 u_1 + \frac{-3/2\, i c{u_1}^2 u_2}{\omega_1} \tag{13}$$

Determinant of $I + D\phi(u)$ is equal to

$$-\frac{1}{64}\,\frac{\left(6\,c{u_2}^2 + 3\,c u_1 u_2 + 6\,c{u_1}^2 - 8\,{\omega_1}^2\right)\left(6\,c{u_2}^2 + 3\,c u_1 u_2 + 6\,c{u_1}^2 + 8\,{\omega_1}^2\right)}{{\omega_1}^4} \tag{14}$$

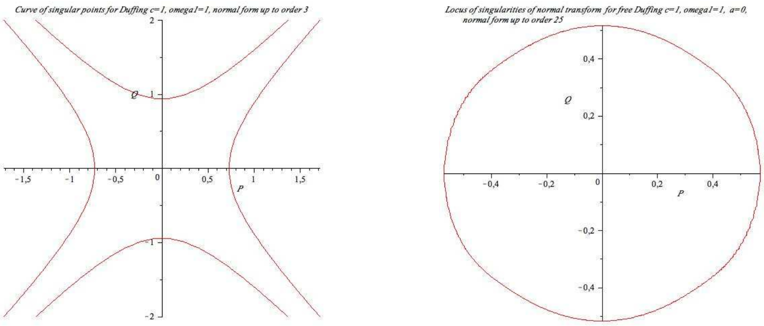

Figure 1. Singularity curve in the plane P, Q where normal coordinate $u_1 = P + iQ$, for $\omega_1 = 1, c = 1$ and normal form calculation up to order 3 (left) and 25 (right).

Since u_1 and u_2 are conjugate, and setting $u_1 = P + iQ$, one obtains in plane P, Q the curve of singular matrix $I + D\phi(u)$ (see figure 1 left). Similar calculations have been done for higher order normal forms and simplest choice of the normal transform (the component of the normal transform according to each resonant term in homological equation is chosen as zero). Details for normal transforms and normal forms are omitted here. We simply provide in the plane P, Q the corresponding results to show that there is a limit ball around $U = 0$ so that $I + D\phi(U)$ is not singular and that the limit radius of this limit ball numerically converges to 0.5 (see figure 1 right for normal form up to degree 25). The maximal convergence radius (maximum of $\mid u_1 \mid$ before the circle centered on $P = Q = 0$ meets the singularity curve) corresponding to successive orders k of normal forms are given in table 1.

Normal form order k	Maximal modulus of u_1
3	0.7303
7	0.5898
17	0.5442
25	0.5170

Table 1. Maximal convergence radius (i.e. maximal $\mid u_1 \mid$ vs order of normal form k).

These numerical results shows that the upper bound of convergence radius tends to $1/2$, which appears reasonable for a perturbation approach: as $w_1 = u_1 + u_2$, it is meaningful to ask $|w_1| \leq 1$ so that $|u_1| = |u_2| \leq 1/2$.

4. A two dofs system

The method is tested on the two dofs system represented in Fig. 2(a) :

$$\ddot{X}_1 + \omega_1^2 X_1 + \frac{\omega_1^2}{2}(3X_1^2 + X_2^2) + \omega_2^2 X_1 X_2 + \frac{\omega_1^2 + \omega_2^2}{2} X_1 (X_1^2 + X_2^2) = 0 \quad (15a)$$

$$\ddot{X}_2 + \omega_2^2 X_2 + \frac{\omega_2^2}{2}(3X_2^2 + X_1^2) + \omega_1^2 X_1 X_2 + \frac{\omega_1^2 + \omega_2^2}{2} X_2 (X_1^2 + X_2^2) = 0 \quad (15b)$$

The real normal form method, up to third order and presented in [3], is used. Figs. 2(b,c)

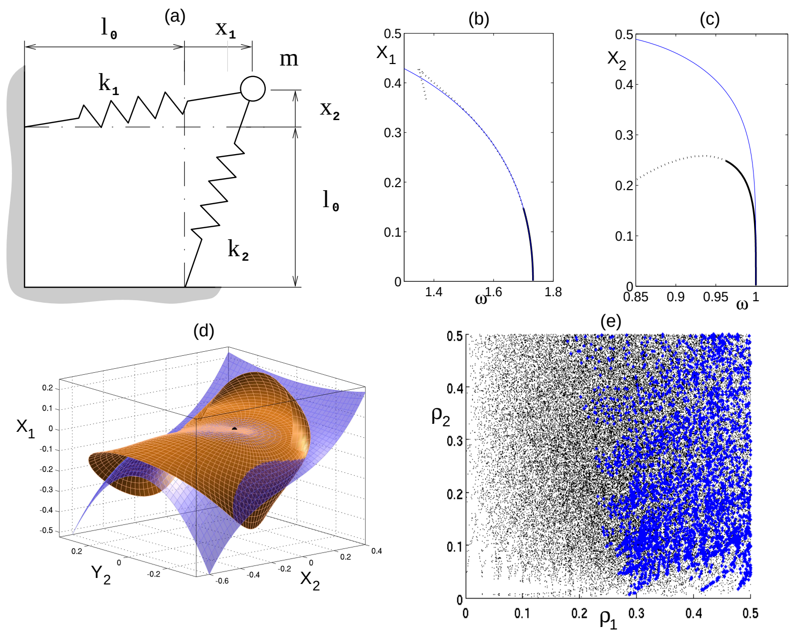

Figure 2. (a): sketch of the two dofs mechanical system. (b)-(c): backbone curves for mode 1 (b) and mode 2 (c), comparison of exact solution (black, numerical solution by continuation) and third-order normal form approach (blue). (d): Second NNM in phase space, blue: third-order approximation, red: exact solution (continuation of periodic orbits). (e): criterion for validity limit in plane (ρ_1, ρ_2). Positive values of Δ in black, negative ones in blue. Red circle of radius ρ=0.26 indicates validity limit.

shows the backbone curves obtained for $\omega_1 = \sqrt{3}$ and $\omega_2 = 1$. The exact solution, obtained numerically by a continuation method, is compared to the backbone curves obtained for third-order approximation provided by the normal form method. One can see that the third-order asymptotic and exact method give coincident results for mode 1 until the sharp bend of the exact solution around $X_1 = 0.43$. On the other hand, the approximation for mode 2 depart from the exact solution around $X_2 = 0.2$. Fig. 2(d) shows the 3-d view of the second NNM for illustration. Once again, third-order approximation is compared to the exact solution obtained by continuation, showing the same divergence of solutions around $X_2 = 0.2$. To assess the upper bound, the determinant $\Delta = \det(I + D\phi(u))$ is computed, where $u = (R_1, S_1, R_2, S_2)$ is now a four-dimensional vector, R_i stands for the normal (real) variable associated to the displacement of NNM i (up to third order) while S_i stands for its velocity. To represent the results, reduced coordinates $\rho_i = \sqrt{X_i^2 + Y_i^2/\omega_i^2}$, i=1,2, are introduced. $\Delta(\rho_1, \rho_2)$ is shown in Fig. 2(e). When ρ_i tends to zero, Δ tends to one, so that in the vicinity of the origin, positive values are found. Negative values are observed in the upper part of the plane, indicating that singular points exist. A red circle of radius ρ=0.26 shows the upper bound obtained for validity limit of third-order normal form approximation, a value that is consistent to the dynamical observation found from the backbone curves.

5. Conclusion

Tracking singular points of the inverse of the jacobian matrix associated to normal form leads to upper bound validity limit for this class of analytical methods. The relevance of the method has been tested on two different examples. A Duffing equation shows that the proposed validity limit tends to 1/2 when increasing the degree k of the normal transformation, which is meaningful for a perturbative scheme. A two dofs system shows that the method can be extended easily to larger systems, and gives a coherent upper bound for a rapid prediction of the validity of the third-order approximation.

References

1. G. Iooss, M. Adelmeyer: *Topics in Bifurcation Theory and Applications*, Singapore, World Scientific, 1992.
2. L. Jézéquel, C.-H. Lamarque: Analysis of Nonlinear Structural Vibrations by Normal Form Theory, *Journal of Sound and Vibration*, 149 (3), 1991, 429-459.
3. C. Touzé, O. Thomas, A. Chaigne: Hardening/softening behaviour in non-linear oscillations of structural systems using non-linear modes, *Journal of Sound and Vibration*, 273 (1-2), 2004, 77-101.
4. A.D. Brjuno: Analytical Forms of Differential Equations, *Trudy Mosk. Mat. Obshest.*, 25, 1971, 119-262 and 25, 1972, 199-299.

Claude-Henri Lamarque, Professor: Université de Lyon, ENTPE, DGCB FRE CNRS 3237, Rue Maurice Audin, 69518 Vaulx-en-Velin Cedex, France, the author gave a presentation of this paper during one of the conference sessions (*lamarque@entpe.fr*).

Cyril Touzé, Assistant Professor: ENSTA- ParisTech, UME, Chemin de la Hunière, 91761 Palaiseau Cedex, France (*cyril.touze@ensta.fr*).

Olivier Thomas, Assistant Professor: CNAM, LMSSC, 2 rue Conté, 75141 Paris Cedex 03, France (*olivier.thomas@cnam.fr*).