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# A Rellich type theorem for the Helmholtz equation in a conical domain

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## Abstract

We prove that there cannot exist square-integrable nonzero solutions to the Helmholtz equation in an axisymmetric conical domain whose vertex angle is greater than  $\pi$ . This implies in particular the absence of eigenvalues embedded in the essential spectrum of a large class of partial differential operators which coincide with the Laplacian in the conical domain.

## 1 Introduction

The purpose of this note is to prove the following result.

**Theorem 1.** *Let  $k > 0$ ,  $\theta \in (0, \pi/2)$  and  $\Omega := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}; y > -|x| \tan \theta\}$  with  $d \geq 1$  (see Figure 1). If  $u \in L^2(\Omega)$  satisfies the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega \tag{1}$$

*in the distributional sense, then  $u = 0$ .*

This theorem is optimal in the sense that it becomes false if  $\theta = 0$ . Indeed it is easy to construct solutions to the Helmholtz equation which are square-integrable in a half-plane (see Remark 5).

As the well-known Rellich's uniqueness theorem [8] and succeeding results (see, e.g., [9]), this theorem points out a forbidden behavior of solutions to the Helmholtz equation in an unbounded domain. In particular, it is not related to boundary conditions (no assumption is made on the behavior of  $u$  near the boundary of  $\Omega$ ). Most Rellich type results involve a particular Besov space related to the boundedness of the energy flux and lead to the uniqueness of the solution to scattering problems. Our theorem involves a more restrictive functional framework: the assumption  $u \in L^2(\Omega)$  rather expresses the boundedness of the total energy in  $\Omega$ , which leads to the absence of so-called *trapped modes* (or *bound states*) or equivalently, the absence of eigenvalues embedded in the continuous spectrum of a large class of operators which coincide with the Laplacian in  $\Omega$ . For instance, if we consider the equation

$$\Delta u + k^2 n^2 u = 0 \quad \text{in } \mathbb{R}^{d+1},$$

with a variable index of refraction  $n = n(x)$  (say, bounded) such that  $n = 1$  in  $\Omega$ , Theorem 1 together with the unique continuation principle (see, e.g., [6]) shows that this equation has no square-integrable nonzero solutions.

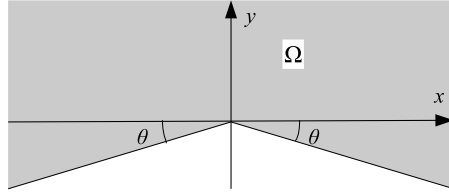


Figure 1: The conical domain  $\Omega$  in the two-dimensional case ( $d = 1$ ).

Our initial motivation concerned the possible existence of *trapped modes* in waveguides. Such solutions are known to occur for local perturbations of *closed* uniform waveguides, that is, cylindrical infinite pipes with bounded cross-section (see [7] for a review). Trapped modes also appear in the case of curved *closed* waveguides (see, e.g., [4] and more recently [3]). The situation differs singularly in the case of *open* waveguides, that is, when the transverse section becomes unbounded, as for instance optical fibers or immersed pipes. It is now understood that trapped modes do not exist in local perturbations of *open* straight waveguides [1, 5], neither in straight junctions of waveguides [2]. The example above shows that this result holds true for curved open waveguides provided that the core of the waveguide and other possible inhomogeneities are located in  $\mathbb{R}^{d+1} \setminus \Omega$ .

The proof of Theorem 1 is based on the two-dimensional case which is explained in sections 2 and 3. Section 4 then shows how to deal with higher dimensions as well as some possible extensions of Theorem 1. Following [1, 2, 5] (inspired by the pioneering work of Weder [10]), the main ingredients of the proof are on the one hand, a *Fourier representation* of a solution  $u$  to (1) in a half-plane and on the other hand, an *analyticity property*. The Fourier representation consists in decomposing  $u$  as a superposition of modes, which are either propagative or evanescent. Since we are only interested in square-integrable solutions, the components associated with propagative modes must vanish. As in the above mentioned papers, the fact that the components associated with evanescent modes also vanish results from the *analyticity property*. Here the key idea to obtain this property is to reuse the Fourier representation in two oblique directions.

## 2 Fourier representation in a half-plane

Let  $\mathcal{F}$  denote the usual Fourier transform defined for a function  $\varphi \in L^1(\mathbb{R})$  by

$$\mathcal{F}\varphi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(x) e^{-ix\xi} dx \quad \text{for } \xi \in \mathbb{R},$$

which can be classically extended to the Schwartz space of tempered distributions.

**Proposition 2.** *For given  $\varepsilon > 0$  and  $k > 0$ , let  $u$  be a solution to the Helmholtz equation (1) in the half-plane  $\Pi_\varepsilon := \mathbb{R} \times (-\varepsilon, +\infty)$  such that  $u \in L^2(\Pi_\varepsilon)$ . Then*

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{|\xi| > k} \widehat{\varphi}(\xi) e^{ix\xi - y\sqrt{\xi^2 - k^2}} d\xi \quad \text{for all } (x, y) \in \overline{\Pi_0}, \quad (2)$$

where  $\widehat{\varphi} := \mathcal{F}u(\cdot, 0)$  has the following properties:

$$\widehat{\varphi}(\xi) = 0 \text{ if } |\xi| < k \quad \text{and} \quad \frac{\widehat{\varphi}(\xi) e^{\varepsilon|\xi|}}{|\xi^2 - k^2|^{1/4}} \in L^2(\mathbb{R}). \quad (3)$$

**Remark 3.** Formula (2) appears as a *modal representation* of  $u$  (analogous to the *plane wave spectrum representation* of Fourier optics) in the sense that it can be interpreted as a superposition of the modes  $\exp(ix\xi - y\sqrt{\xi^2 - k^2})$  where  $\widehat{\varphi}(\xi)$  stands for the modal amplitude. These modes are propagative in the  $x$ -direction and evanescent in the  $y$ -direction, since  $|\xi| > k$  in (2). The first property in (3) expresses actually the absence of propagative modes in the  $y$ -direction, which results from the assumption  $u \in L^2(\Pi_\varepsilon)$ .

**Remark 4.** The domain  $\Pi_\epsilon$  in which  $u$  is supposed to satisfy the Helmholtz equation is larger than the domain  $\Pi_0$  where the representation (2) is written. This allows us to avoid the consequences of a possible poor regularity of  $u$  at the boundary of  $\Pi_\epsilon$  and yields a strong decay of  $\widehat{\varphi}(\xi)$  as  $|\xi|$  tends to  $\infty$ , which is expressed by the exponential term in the second property of (3). Note that by Cauchy–Schwarz inequality, this property implies that  $\widehat{\varphi} \in L^1(\mathbb{R})$ , which shows that the integral in (2) makes sense.

**Remark 5.** Using the arguments of the proof below, it is readily seen that conversely to the statement of Proposition 2, if  $\widehat{\varphi}$  satisfies (3), then the function  $u$  defined by (2) belongs to  $L^2(\Pi_0)$  and is a solution to the Helmholtz equation (1) in  $\Pi_0$ . This shows that Theorem 1 is no longer true for  $\theta = 0$ .

*Proof.* As  $u \in L^2(\Pi_\epsilon)$ , we know that for almost every  $Y \in (-\epsilon, +\infty)$ , function  $u(\cdot, Y)$  belongs to  $L^2(\mathbb{R})$  so that we can define its Fourier transform  $\widehat{u}(\xi, Y) := \mathcal{F}\{u(\cdot, Y)\}(\xi)$ . By the Parseval’s identity,

$$\widehat{u}(\cdot, Y) \in L^2(\mathbb{R}) \quad \text{and} \quad \|\widehat{u}(\cdot, Y)\|_{L^2(\mathbb{R})} = \|u(\cdot, Y)\|_{L^2(\mathbb{R})}.$$

As a consequence

$$\widehat{u} \in L^2(\mathbb{R} \times (-\epsilon, +\infty)) \quad \text{and} \quad \|\widehat{u}\|_{L^2(\mathbb{R} \times (-\epsilon, +\infty))} = \|u\|_{L^2(\Pi_\epsilon)}. \quad (4)$$

Applying the Fourier transform to the Helmholtz equation yields

$$\frac{\partial^2 \widehat{u}}{\partial Y^2} + (k^2 - \xi^2) \widehat{u} = 0 \quad \text{in } \mathcal{D}'(-\epsilon, +\infty) \quad \text{for a.e. } \xi \in \mathbb{R}.$$

Hence

$$\widehat{u}(\xi, Y) = \widehat{A}(\xi) e^{-(Y+\epsilon)\sqrt{\xi^2-k^2}} + \widehat{B}(\xi) e^{+(Y+\epsilon)\sqrt{\xi^2-k^2}},$$

for some functions  $\widehat{A}(\xi)$  and  $\widehat{B}(\xi)$ , where  $\sqrt{\cdot}$  denotes a given determination of the complex square root such that  $\sqrt{z} \in \mathbb{R}^+$  if  $z \in \mathbb{R}^+$ . From (4), we have  $\widehat{u}(\xi, \cdot) \in L^2(-\epsilon, +\infty)$  for almost every  $\xi \in \mathbb{R}$ , which implies that on the one hand,  $\widehat{A}(\xi) = \widehat{B}(\xi) = 0$  if  $|\xi| < k$ , on the other hand,  $\widehat{B}(\xi) = 0$  if  $|\xi| > k$ . Therefore

$$\widehat{u}(\xi, Y) = \widehat{A}(\xi) e^{-(Y+\epsilon)\sqrt{\xi^2-k^2}} \quad \text{where} \quad \widehat{A}(\xi) = 0 \quad \text{if } |\xi| < k. \quad (5)$$

Noticing that

$$\|\widehat{u}\|_{L^2(\mathbb{R} \times (-\epsilon, +\infty))}^2 = \int_{|\xi|>k} |\widehat{A}(\xi)|^2 \int_{-\epsilon}^{+\infty} e^{-2(Y+\epsilon)\sqrt{\xi^2-k^2}} dY d\xi = \int_{|\xi|>k} |\widehat{A}(\xi)|^2 \frac{d\xi}{2\sqrt{\xi^2-k^2}},$$

we infer that  $|\xi^2 - k^2|^{-1/4} \widehat{A}(\xi) \in L^2(\mathbb{R})$ . Setting  $\widehat{\varphi}(\xi) := \widehat{A}(\xi) e^{-\epsilon\sqrt{\xi^2-k^2}}$  and using the inverse Fourier transform of (5), the conclusion follows.  $\square$

The following corollary plays an essential role in the proof of Theorem 1.

**Corollary 6.** For any half-line  $\Lambda_\alpha := \{(t \cos \alpha, t \sin \alpha) \in \mathbb{R}^2; t > 0\}$  where  $\alpha \in (0, \pi)$ , a solution  $u \in L^2(\Pi_\epsilon)$  to the Helmholtz equation (1) in  $\Pi_\epsilon$  is such that  $u|_{\Lambda_\alpha} \in L^1(\Lambda_\alpha)$ .

*Proof.* By the Fourier representation (2) of  $u$ , we have

$$\begin{aligned} \int_0^{+\infty} |u(t \cos \alpha, t \sin \alpha)| dt &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \left| \int_{|\xi|>k} \widehat{\varphi}(\xi) e^{i\xi t \cos \alpha - \sqrt{\xi^2-k^2} t \sin \alpha} d\xi \right| dt \\ &\leq \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \int_{|\xi|>k} |\widehat{\varphi}(\xi)| e^{-\sqrt{\xi^2-k^2} t \sin \alpha} d\xi dt \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{|\xi|>k} \frac{|\widehat{\varphi}(\xi)|}{\sqrt{\xi^2-k^2} \sin \alpha} d\xi, \end{aligned}$$

using Fubini’s theorem. We deduce from the Cauchy–Schwarz inequality that

$$\int_0^{+\infty} |u(t \cos \alpha, t \sin \alpha)| dt \leq \frac{1}{\sqrt{2\pi}} \left\| \frac{\widehat{\varphi}(\xi) e^{\epsilon|\xi|}}{|\xi^2 - k^2|^{1/4}} \right\|_{L^2(\mathbb{R})} \left( \int_{|\xi|>k} \frac{e^{-2\epsilon|\xi|}}{\sqrt{\xi^2 - k^2} \sin^2 \alpha} d\xi \right)^{1/2},$$

where the right-hand side is bounded, according to (3).  $\square$

### 3 Proof of Theorem 1 in the two-dimensional case ( $d = 1$ )

The proof is based on three uses of Proposition 2. In order to simplify the presentation as regards the role of  $\varepsilon$  in this proposition, we redefine the domain  $\Omega$  of Theorem 1 as  $\Omega := \{(x, y) \in \mathbb{R}^2; y + \ell > -|x| \tan \theta\}$  for some  $\ell > 0$  (which amounts to a simple change of variable).

In the first use of Proposition 2, we simply notice that  $u$  is a square-integrable solution to the Helmholtz equation in the half-plane  $\{y > -\ell\}$ . Hence, the conclusions of the proposition hold true with  $(X, Y) = (x, y)$  and  $\varepsilon = \ell$ . In particular,

$$\widehat{\varphi} := \mathcal{F}u(\cdot, 0) \text{ is such that } \widehat{\varphi}(\xi) = 0 \text{ if } |\xi| < k. \quad (6)$$

The key argument consists in proving that  $\widehat{\varphi}(\xi)$  extends to an analytic function in a complex vicinity of the real axis. As  $\widehat{\varphi}$  vanishes on the interval  $(-k, +k)$ , analyticity implies that it must vanish everywhere, that is,  $\widehat{\varphi}(\xi) = 0$  for all  $\xi$  (recall that the zeros of an analytic function are isolated). The Fourier representation (2) then tells us that  $u$  vanishes in the half-plane  $\{y > 0\}$ , so also in the whole domain  $\Omega$  by virtue of the unique continuation principle, which completes the proof of Theorem 1.

It remains to prove the analyticity of  $\widehat{\varphi}(\xi)$ . To do this, the trick is to split the definition  $\widehat{\varphi} := \mathcal{F}u(\cdot, 0)$  in the form

$$\widehat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 u(x, 0) e^{-ix\xi} dx + \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} u(x, 0) e^{-ix\xi} dx \quad (7)$$

and to express  $u(\cdot, 0)|_{\mathbb{R}^+}$  and  $u(\cdot, 0)|_{\mathbb{R}^-}$  by using again Proposition 2 in two half-planes contained in  $\Omega$  defined respectively by the equations  $y > -x \tan \theta$  and  $y > +x \tan \theta$ . Using both changes of variables

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos \theta & \mp \sin \theta \\ \pm \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which map respectively the half-planes  $\{y > \mp x \tan \theta\}$  to  $\Pi_0$ , we obtain the following Fourier representations:

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{|\eta| > k} \widehat{\varphi}^\pm(\eta) e^{i\eta(x \cos \theta \mp y \sin \theta) - \sqrt{\eta^2 - k^2}(\pm x \sin \theta + y \cos \theta)} d\eta \quad \text{if } y > \mp x \tan \theta,$$

where  $\widehat{\varphi}^\pm$  both satisfy (3) with  $\varepsilon = \ell \cos \theta$ . Taking  $y = 0$  yields  $u(\cdot, 0)|_{\mathbb{R}^\pm}$ . Note that Corollary 6 tells us that  $u(\cdot, 0)|_{\mathbb{R}^\pm} \in L^1(\mathbb{R}^\pm)$ . This justifies equation (7) which becomes

$$\widehat{\varphi}(\xi) = \frac{1}{2\pi} \sum_{\pm} \int_{\mathbb{R}^\pm} \int_{|\eta| > k} \widehat{\varphi}^\pm(\eta) e^{x(i\eta \cos \theta \mp \sqrt{\eta^2 - k^2} \sin \theta)} d\eta e^{-ix\xi} dx.$$

Using the same argument as in the proof of Corollary 6, it is readily seen that the integrands of the above integrals belong respectively to  $L^1(\mathbb{R}^\pm \times \{|\eta| > k\})$ . Hence, Fubini's theorem yields

$$\widehat{\varphi}(\xi) = \frac{1}{2\pi} \sum_{\pm} \int_{|\eta| > k} \widehat{\varphi}^\pm(\eta) \int_{\mathbb{R}^\pm} e^{x(i(\eta \cos \theta - \xi) \mp \sqrt{\eta^2 - k^2} \sin \theta)} dx d\eta = \int_{|\eta| > k} F(\eta, \xi) d\eta,$$

where

$$F(\eta, \xi) := \frac{1}{2\pi} \sum_{\pm} \frac{\widehat{\varphi}^\pm(\eta)}{\mp i(\eta \cos \theta - \xi) + \sqrt{\eta^2 - k^2} \sin \theta}.$$

For almost every  $\eta$ , this function extends to an analytic function of  $\xi$  in any complex domain in which both denominators do not vanish. The complex values of  $\xi$  for which there exists a  $\eta \in \mathbb{R} \setminus (-k, +k)$  such that one of the denominators vanishes is the hyperbola defined by the equation

$$\frac{(\operatorname{Re} \xi)^2}{\cos^2 \theta} - \frac{(\operatorname{Im} \xi)^2}{\sin^2 \theta} = k^2$$

represented in Figure 2. Hence, for almost every  $\eta$ ,  $F(\eta, \xi)$  is an analytic function of  $\xi$  in the three connected components of the complex plane delimited by this hyperbola. By the Lebesgue's dominated convergence theorem (recall that  $\widehat{\varphi}^\pm \in L^1(\mathbb{R})$ , see Remark 4), we deduce that the same holds true for  $\widehat{\varphi}(\xi)$ . In fact, we are interested in the analyticity of  $\widehat{\varphi}(\xi)$  in the components colored in gray in Figure 2. Indeed, as we already know that  $\widehat{\varphi}$  vanishes on  $(-k, +k)$  (see (6)), the analyticity in these gray components implies that  $\widehat{\varphi}(\xi)$  also vanishes for  $\xi \in \mathbb{R} \setminus (-k, +k)$ , which is the desired result.

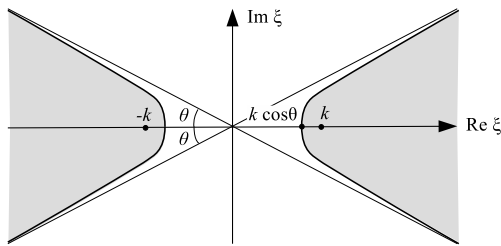


Figure 2: Domains of analyticity of  $\hat{\varphi}$

## 4 Higher dimensions and other extensions

The above proof of Theorem 1 is specific to the two-dimensional case, since it cannot be extended directly to higher dimensions (more precisely, Proposition 2 can readily be extended, but not the arguments of section 3). Fortunately, the case  $d > 1$  is easily derived from the case  $d = 1$  as follows. Suppose that  $u \in L^2(\Omega)$  satisfies the Helmholtz equation (1) in  $\Omega$ . We split the coordinate  $x \in \mathbb{R}^d$  as  $x = (x_1, x')$  where  $x' := (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$  so that the Fourier transform  $\mathcal{F}$  in the  $x$  variable appears as the product of the partial Fourier transforms in the  $x_1$  and  $x'$  variables, denoted by  $\mathcal{F}_1$  and  $\mathcal{F}'$  respectively. Define  $u'(x_1, \xi', y) := \mathcal{F}'\{u(x_1, \cdot, y)\}(\xi')$  and  $\hat{u}(\xi, y) := \mathcal{F}\{u(\cdot, y)\}(\xi)$  where  $\xi' \in \mathbb{R}^{d-1}$  and  $\xi = (\xi_1, \xi') \in \mathbb{R}^d$  are the Fourier variables associated respectively with  $x'$  and  $x$ . By the Parseval's identity, for a.e.  $\xi' \in \mathbb{R}^{d-1}$ ,  $u'(\cdot, \xi', \cdot)$  is square-integrable in the cone  $\Omega_{2D} := \{(x_1, y) \in \mathbb{R}^2; y > -|x_1| \tan \theta\}$  and satisfies

$$\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y^2} + k^2 - |\xi'|^2 \right) u'(\cdot, \xi', \cdot) = 0 \quad \text{in } \Omega_{2D}.$$

Hence Theorem 1 for  $d = 1$  applies provided  $|\xi'| < k$ : in this case, we thus know that  $u'(\cdot, \xi', \cdot)$  vanishes in  $\Omega_{2D}$ . As  $\hat{u}(\xi_1, \xi', y) = \mathcal{F}_1\{u'(\cdot, \xi', y)\}(\xi_1)$ , this shows that for all  $y \geq 0$ ,  $\hat{u}(\cdot, y)$  vanishes in the cylinder  $\{(\xi_1, \xi') \in \mathbb{R}^d; |\xi'| < k\}$ . This conclusion actually remains true for any axisymmetric cylinder of radius  $k$  whose axis contains the origin, since in the above lines, one can replace the particular direction  $x_1$  by any direction of the  $x$ -space using a rotation. Therefore  $\hat{u}(\xi, y) = 0$  for all  $\xi \in \mathbb{R}^d$  and  $y \geq 0$ . Applying  $\mathcal{F}^{-1}$  yields  $u = 0$  in  $\mathbb{R}^d \times \mathbb{R}^+$  and we conclude again by the unique continuation principle.

Theorem 1 also applies in an anisotropic medium described by the equation

$$\operatorname{div}(\mathbb{A} \operatorname{grad} u) + k^2 u = 0 \quad \text{in } \Omega,$$

where  $\mathbb{A}$  is a constant real symmetric positive definite  $d \times d$  matrix. Indeed this equation can be transformed into our original Helmholtz equation using first the new coordinate system corresponding to a unitary matrix which diagonalizes  $\mathbb{A}$ , then a suitable dilation in each direction. This process transforms  $\Omega$  into a new conical domain that is no longer axisymmetric in general, but that still contains an axisymmetric cone with vertex angle greater than  $\pi$ , which allows us to conclude.

Another application concerns the time-harmonic Maxwell's equations

$$\operatorname{curl} \operatorname{curl} \mathbf{U} - k^2 \mathbf{U} = 0 \quad \text{in } \Omega \subset \mathbb{R}^3.$$

In this case, we simply have to notice that, thanks to the relation  $\Delta = -\operatorname{curl} \operatorname{curl} + \operatorname{grad} \operatorname{div}$ , if  $\mathbf{U} \in L^2(\Omega)^3$  satisfies this equation, then each of its components  $\mathbf{U}_i$  belongs to  $L^2(\Omega)$  and satisfies the Helmholtz equation (1).

Theorem 1 can also be extended to some situations which involve non-homogeneous media, using a generalized Fourier transform instead of the usual one, as shown in [1, 2, 5]. Works on this subject are in progress.

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