Probabilistic representation of a class of non conservative nonlinear Partial Differential Equations

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Short title. About probabilistic representation of non-conservative PDEs.

Abstract

We introduce a new class of nonlinear Stochastic Differential Equations in the sense of McKean, related to non-conservative nonlinear Partial Differential equations (PDEs). We discuss existence and uniqueness pathwise and in law under various assumptions.

Key words and phrases: Nonlinear Partial Differential Equations; Nonlinear McKean type Stochastic Differential Equations; Probabilistic representation of PDEs; Wasserstein type distance.

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1 Introduction

Probabilistic representations of nonlinear Partial Differential Equations (PDEs) are interesting in several aspects. From a theoretical point of view, such representations allow for probabilistic tools to study the analytical properties of the equation (existence and/or uniqueness of a solution, regularity,...). They also have their own interest, typically when they provide a microscopic interpretation of physical phenomena macroscopically modeled by a nonlinear PDE. In addition, from a numerical point of view, such representations allow for new approximation schemes potentially less sensitive to the dimension of the state space thanks to their probabilistic nature involving Monte Carlo based methods.

The paper focuses on a specific forward approach. The underlying idea of our paper consists in extending to fairly general PDEs the probabilistic representation of non-linear Fokker-Planck equations, which are conservative. The probabilistic representation is based on a generalized nonlinear stochastic differential equation (SDE) in the sense of McKean [15], whose coefficients do not depend only on the position of the solution Y, but also on the law of the process Y. In the companion paper [13], we will investigate the associated interacting particle systems, the propagation of chaos with related numerical aspects.

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Let us consider $d, p \in \mathbb{N}^\star$. Let $\Phi : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d\times p}$, $g : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$, $\Lambda : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, be Borel bounded functions and ζ_0 be a probability on $\mathbb{R}^d$. When it is absolutely continuous v_0 will designate its density so that $\zeta_0(dx) = v_0(x)dx$. The target of the present paper is a non-linear PDE (in the sense of distributions) of the form

$$\left\{ \begin{array}{l} \partial_t v = \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij}^2 \left((\Phi\Phi^t)_{i,j}(t,x,v)v \right) - div\,(g(t,x,v)v) + \Lambda(t,x,v)v \ , \quad \text{for any } t \in [0,T] \ , \\ v(0,dx) = \zeta_0(dx), \end{array} \right. \tag{1.1}$$

where $v :]0,T] \times \mathbb{R}^d \to \mathbb{R}$ is the unknown function and the second equation means that $v(t,x)dx$ converges weakly to $\zeta_0(dx)$ when $t \to 0$. In our spirit this should constitute the first step of investigation of a class of PDEs where the coefficients Φ, g, Λ may also depend on some space derivatives of v.

When $\Lambda = 0$, PDEs of the type (1.1) are non-linear generalizations of the Fokker-Planck equation. In that case, solutions v of (1.1) are in general *conservative* in the sense that $\int_{\mathbb{R}} v(t,x)dx$ is constant in t, so equal to 1 if the initial condition is a probability measure. In particular when Φ and g do not depend on v, then previous equation is a classical (time-dependent) *Fokker-Planck type equation*. Under reasonnable conditions on Φ and g (for instance if they are Lipschitz with linear growth or bounded continuous), then according to Theorem 5.1.1 and Corollary 6.4.4 of [17], there is a process Y which is a solution, at least in law (for any initial condition) to a SDE with diffusion (resp. drift) coefficient equal to Φ (resp. g). Indeed that solution does not explode. So Itô's formula applied to $\varphi(Y)$, where φ is a test function, allows to show that the function ν defined on $[0,T]$ with values in the space of finite measures such that ν_t is the marginal law of Y_t, is a solution of (1.1). This shows in particular that ν is conservative. Coming back to the non-linear case, i.e. when Φ, g may depend on v, once a solution v of (1.1) (in the sense of distributions) is known, using approximation arguments it is not difficult to show that v is conservative, at least when $(s,x) \mapsto g(s,x,v(s,x))v(s,x)$ and $(s,x) \mapsto (\Phi\Phi^t)_{i,j}(s,x,v(s,x))v(s,x)$ are integrable functions. An important particular case of (1.1) is given by the case when $g = 0$ and $\Phi(t,x,v) = \tilde{\Phi}(v)I_d$, where I_d is the identity matrix on $\mathbb{R}^d$ and $\tilde{\Phi} : \mathbb{R} \to \mathbb{R}_+$. When $\tilde{\Phi}(v) = |v|^{\frac{m-1}{2}}$ for $m > 1$ (resp. $0 < m < 1$), (1.1) is called porous media (resp. fast diffusion) equation. In that case explicit solutions exist, the so called **Barenblatt** solutions. If those solutions are all conservative in the case of porous media, that property can be lost in the case of fast diffusion, see chapter 9 of [20] or more in details [19], at least when $m < \frac{d-1}{d}$.

To summarize, if $\Lambda = 0$, in the conservative case, starting from a probability measure ζ_0 as initial condition, the solutions of (1.1) are probability measures dynamics which often describe the macroscopic distribution law of a *microscopic particle* which behaves in a diffusive way. More precisely, often, the solution v of (1.1) is associated with a couple (Y,v), where Y is a stochastic process and v a real valued function defined on $[0,T] \times \mathbb{R}^d$ such that

$$\left\{ \begin{array}{l} Y_t = Y_0 + \int_0^t \Phi(s,Y_s,v(s,Y_s))dW_s + \int_0^t g(s,Y_s,v(s,Y_s))ds \ , \quad \text{with } Y_0 \sim \zeta_0 \\ v(t,\cdot) \text{ is the density of the law of } Y_t \ , \end{array} \right. \tag{1.2}$$

and $(W_t)_{t\geq 0}$ is a p-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. A major technical difficulty arising when studying the existence and uniqueness for solutions of (1.2) is due to the point dependence of the SDE coefficients w.r.t. the probability density v. In the literature, (1.2) was generally faced by analytical methods. A lot of work was performed in the case of smooth Lipschitz coefficients with regular initial condition, see for instance Proposition 1.3 of [11]. The authors also assumed to be in the non-degenerate case, with $\Phi\Phi^t$ being an invertible matrix with related parabolicity condition. In dimension $d = 1$, an important earlier work concerns the case of porous media equation, see e.g. [6]. Still in dimension

$d = 1$, with $g = 0$ and Φ being bounded measurable, probabilistic representations of (1.1) via solutions of (1.2) were obtained in [8, 1]. [4] extends partially those results to the multidimensional case. Finally [5] treated the case of fast diffusion. All those techniques were based on the resolution of the corresponding non-linear Fokker-Planck equation.

In the present article, we are however especially interested in (1.1), in the case where Λ does not vanish. In that context, the natural generalization of (1.2) is given by

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s, v(s, Y_s))dW_s + \int_0^t g(s, Y_s, v(s, Y_s))ds \ , \quad \text{with} \quad Y_0 \sim \zeta_0 \ , \\ v(t, \cdot) := \frac{d\nu_t}{dx} \quad \text{such that for any bounded continuous test function } \varphi \in \mathcal{C}_b(\mathbb{R}^d) \\ \nu_t(\varphi) := \mathbb{E}\left[\varphi(Y_t) \, \exp\left\{\int_0^t \Lambda\big(s, Y_s, v(s, Y_s)\big)ds\right\}\right] \ , \quad \text{for any } t \in [0, T] \ . \end{cases} \tag{1.3}$$

The aim of the paper is precisely to extend the McKean probabilistic representation to a large class of nonconservative PDEs. The first step in that direction was done by [2] where the Fokker-Planck equation is a stochastic PDE with multiplicative noise. Even though that equation is pathwise not conservative, the expectation of the mass was constant and equal to 1. Here again, these developments relied on analytical tools.

To avoid the technical difficulty due to the pointwise dependence of the SDE coefficients w.r.t. the function v, this paper focuses on the following regularized version of (1.3):

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s, u(s, Y_s))dW_s + \int_0^t g(s, Y_s, u(s, Y_s))ds \ , \quad \text{with} \quad Y_0 \sim \zeta_0 \ , \\ u(t, y) = \mathbb{E}[K(y - Y_t) \, \exp\left\{\int_0^t \Lambda\big(s, Y_s, u(s, Y_s)\big)ds\right\}] \ , \quad \text{for any } t \in [0, T] \ , \end{cases} \tag{1.4}$$

where $K : \mathbb{R}^d \to \mathbb{R}$ is a mollifier in $\mathbb{R}^d$. One historical contribution on the subject in the conservative case $\Lambda = 0$, based on probabilistic methods, was performed by [18], which concentrated on non-linearities only on the drift coefficients. When $K = \delta_0$ (1.4) reduces, at least formally to (1.3).

An easy application of Itô's formula (see e.g. Theorem 6.1) shows that if there is a solution (Y, u) of (1.4), then u is related to the solution (in the distributional sense) of the following partial integro-differential equation (PIDE)

$$\begin{cases} \partial_t \bar{v} = \frac{1}{2} \sum_{i,j=1}^{d} \partial^2_{ij} \big((\Phi\Phi^t)_{i,j}(t, x, K * \bar{v})\bar{v}\big) - div\,(g(t, x, K * \bar{v})\bar{v}) + \Lambda(t, x, K * \bar{v})\bar{v} \\ \bar{v}(0, x) = v_0 \ , \end{cases} \tag{1.5}$$

by the relation $u = K * \bar{v} := \int_{\mathbb{R}^d} K(\cdot - y)\bar{v}(y)dy$. Setting $K^\varepsilon(x) = \frac{1}{\varepsilon^d} K^1\left(\frac{\cdot}{\varepsilon}\right)$ the generalized sequence K^ε is weakly convergent to the Dirac measure at zero. Now, consider the couple $(Y^\varepsilon, u^\varepsilon)$ solving (1.4) replacing K with K^ε. Ideally, u^ε should converge to a solution of the *limit partial differential equation* (1.1). In the case $\Lambda = 0$, with smooth Φ, g and initial condition with other technical conditions, that convergence was established in Lemma 2.6 of [11]. In our extended setting ($\Lambda \neq 0$), again, no mathematical argument is for the moment available but this limiting behavior is explored empirically by numerical simulations in Section 3. of [13]. A convergence analysis has been however performed by the authors in [14], in the particular case when Φ and g do not depend on $\bar{v}$.

The main contribution of this paper comes from a refined analysis of existence and/or uniqueness of a solution to (1.4) under a variety of regularity assumptions on the coefficients Φ, g and Λ. The system (1.4), whose unknown is a couple (Y, u) where Y is a process and u is a function, is composed by two equations. The first one is a stochastic differential equation whose coefficients depend on u and the second equation links u to the law of Y in a non-anticipating way. In the classical McKean type equations, $u(t, \cdot)$

was explicitly provided by the density of the (marginal) law of $Y_t, t > 0$. This situation can be recovered formally here when the function $\Lambda = 0$ and the mollifier $K = \delta_0$. The second equation of (1.4), which involves Λ as a *weighting function*, is indeed the central object of the analysis. $u(t, \cdot)$ is now implicitly related to the the law m^Y of the whole path of process Y. That equation associates to a probability law m on $\mathcal{C}^d := \mathcal{C}([0,T], \mathbb{R}^d)$, a real-valued function u. A significant contribution of the paper consists in analyzing the regularity properties of this relation.

In Section 3, one shows existence and uniqueness of strong solutions of (1.4) when Φ, g, Λ are Lipschitz. This result is stated in Theorem 3.9. The second equation of (1.4) can be rewritten as

$$u(t,y) = \int_{\mathcal{C}^d} K(y - \omega_t) \exp\left\{ \int_0^t \Lambda\big(s, \omega_s, u(s, \omega_s)\big) ds \right\} dm(\omega) \, , \tag{1.6}$$

where $m = m_Y$ is the law of Y on the canonical space $\mathcal{C}^d$. In particular, given a law m on $\mathcal{C}^d$, using an original fixed point argument on stochastic processes Z of the type $Z_t = u(t, X_t)$, where X is the canonical process, in Theorem 3.1, we first study the existence of $u = u^m$ being solution of (1.6). Proposition 3.3 focuses on the analysis of the functional $(t, x, m) \mapsto u^m(t, x)$: this associates to each Borel probability measure m on $\mathcal{C}^d$, the solution of of (1.6). In particular that proposition describes carefully the dependence on all variables. The study of the first equation in (1.4) is based on more standard arguments following Sznitman [18]. The rest of the paper is organized as follows. In Section 4, we show strong existence of (1.4) when Φ, g are Lipschitz and Λ is only continuous, see Theorem 4.2. Indeed, uniqueness, however, does not hold if Λ is only continuous, see Example 4.1. In Section 5, Theorem 5.1 states existence in law in all cases when Φ, g, Λ are only continuous. Section 6 establishes the link between (1.4) and the integro-partial-differential equation (1.5).

2 Notations and assumptions

For any Polish space E, $\mathcal{B}(E)$ will denote its Borel σ-field. It is well-known that the space of Borel probability measures on E, $\mathcal{P}(E)$ is also a Polish space with respect to the weak convergence topology, whose Borel σ-field will be denoted by $\mathcal{B}(\mathcal{P}(E))$. See Proposition 7.20 and Proposition 7.23, Section 7.4 Chapter 7 in [7] and Theorem 8.3.2 and Theorem 8.9.4 in [9].

Let us consider $\mathcal{C}^d := \mathcal{C}([0,T], \mathbb{R}^d)$ metrized by the supremum norm $\| \cdot \|_\infty$. X will be the canonical process on $\mathcal{C}^d$. For $t \geq 0$ we also denote $\mathcal{B}_t(\mathcal{C}^d) := \sigma(X_u, 0 \leq u \leq t)$. Given $r \geq 0$, $\mathcal{P}_r(\mathcal{C}^d)$ will denote the set of Borel probability measures on $\mathcal{C}^d$ admitting a moment of order r. For $r = 0$, $\mathcal{P}(\mathcal{C}^d) := \mathcal{P}_0(\mathcal{C}^d)$. When $d = 1$, we often omit it and we simply set $\mathcal{C} := \mathcal{C}^1$.
We recall that the *Wasserstein distance of order* (resp. the *modified Wasserstein distance of order*) r for $r \geq 1$, denoted by $\mathcal{W}_T^r(m, m')$ (resp. $\widetilde{\mathcal{W}}_T^r(m, m')$), for any m and m' in $\mathcal{P}_r(\mathcal{C}^d)$, (resp. $\mathcal{P}(\mathcal{C}^d)$), are such that

$$(\mathcal{W}_t^r(m, m'))^r \quad := \quad \inf_{\mu \in \Pi(m,m')} \left\{ \int_{\mathcal{C}^d \times \mathcal{C}^d} \sup_{0 \leq s \leq t} |X_s(\omega) - X_s(\omega')|^r d\mu(\omega, \omega') \right\}, \; t \in [0,T] \, , \tag{2.1}$$

$$(\widetilde{\mathcal{W}}_t^r(m, m'))^r \quad := \quad \inf_{\mu \in \widetilde{\Pi}(m,m')} \left\{ \int_{\mathcal{C}^d \times \mathcal{C}^d} \sup_{0 \leq s \leq t} |X_s(\omega) - X_s(\omega')|^r \wedge 1 \; d\mu(\omega, \omega') \right\}, \; t \in [0,T] \, , \tag{2.2}$$

where $\Pi(m, m')$ (resp. $\widetilde{\Pi}(m, m')$) denotes the set of Borel probability measures in $\mathcal{P}(\mathcal{C}^d \times \mathcal{C}^d)$ with fixed marginals m and m' belonging to $\mathcal{P}_r(\mathcal{C}^d)$ (resp. $\mathcal{P}(\mathcal{C}^d)$). In this paper we will use very frequently the Wasserstein distances of order 2. For that reason, we will simply use the convention $\mathcal{W}_t := \mathcal{W}_t^2$ (resp. $\widetilde{\mathcal{W}}_t := \widetilde{\mathcal{W}}_t^2$).

Given $N \in \mathbb{N}^\star$, $l \in \mathcal{C}^d$, $l^1, \cdots, l^N \in \mathcal{C}^d$, a significant role in this paper will be played by the Borel measures on $\mathcal{C}^d$ given by δ_l and $\frac{1}{N}\sum_{j=1}^N \delta_{l^j}$.

Remark 2.1. *Given $l^1, \cdots, l^N, \tilde{l}^1, \cdots, \tilde{l}^N \in \mathcal{C}^d$, by definition of the Wasserstein distance we have, for all $t \in [0,T]$,*

$$\mathcal{W}_t\left(\frac{1}{N}\sum_{j=1}^N \delta_{l^j}, \frac{1}{N}\sum_{j=1}^N \delta_{\tilde{l}^j}\right) \le \frac{1}{N}\sum_{j=1}^N \sup_{0\le s\le t} |l_s^j - \tilde{l}_s^j|^2 \,.$$

In this paper $\mathcal{C}_b(\mathcal{C}^d)$ denotes the space of bounded, continuous real-valued functions on $\mathcal{C}^d$. $\mathbb{R}^d$ is equipped with the scalar product $\cdot$ and $|x|$ stands for the induced Euclidean norm for $x \in \mathbb{R}^d$. Given two reals a, b, in the sequel we will adopt the notations $a \wedge b := \min(a,b)$ and $a \vee b := \max(a,b)$.
$\mathcal{S}(\mathbb{R}^d)$ is the space of Schwartz fast decreasing test functions and $\mathcal{S}'(\mathbb{R}^d)$ is its dual. $\mathcal{C}_b(\mathbb{R}^d)$ is the space of bounded, continuous real functions on $\mathbb{R}^d$. $\mathcal{C}_0(\mathbb{R}^d)$ (resp. $\mathcal{C}_0^\infty(\mathbb{R}^d)$) represents the space of real continuous (resp. smooth) functions with compact support in $\mathbb{R}^d$. Given a real (possibly signed) Borel measure μ in $\mathbb{R}^d$, we will denote by $\varphi \mapsto \mu(\varphi)$ or by $\varphi \mapsto \langle \mu, \varphi\rangle$ the duality mapping, where $\varphi \in C_b(\mathbb{R}^d)$. $W^{r,p}(\mathbb{R}^d)$ is the Sobolev space of order $r \in \mathbb{N}, 1 \le p \le \infty$. When $r = 0$ this equals $(L^p(\mathbb{R}^d), ||\cdot||_p)$. $||\cdot||_p$ will also denote the standard norm related to $(L^p(\mathbb{R}^d; E)$ where E is another finite dimensional space. $(\phi_n^d)_{n\ge 0}$ will denote an usual sequence of mollifiers $\phi_n^d(x) = \frac{1}{\epsilon_n^d}\phi^d(\frac{x}{\epsilon_n})$ where, ϕ^d is a non-negative function, belonging to the Schwartz space whose integral is 1 and $(\epsilon_n)_{n\ge 0}$ is a sequence of strictly positive reals verifying $\epsilon_n \xrightarrow[n\longrightarrow\infty]{} 0$.
When $d = 1$, we will simply write $\phi_n := \phi_n^1, \phi := \phi^1$.
$\mathcal{F}(\cdot) : f \in \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^d)$ will be the Fourier transform on the classical Schwartz space $\mathcal{S}(\mathbb{R}^d)$ such that for all $\xi \in \mathbb{R}^d$,

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}dx \,.$$

We will indicate in the same manner the corresponding Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$.
A function $F : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ will be said **uniformly continuous with respect to** (y,z) (the *space variables*) in a subset B of $\mathbb{R}^d \times \mathbb{R}$ **uniformly in** $t \in [0,T]$ if for every $\varepsilon > 0$, there is $\delta > 0$, such that $\forall (y,z), (y',z') \in B$,

$$|y-y'| + |z-z'| \le \delta \Longrightarrow \forall t \in [0,T],\ |F(t,y,z) - F(t,y',z')| \le \varepsilon. \tag{2.3}$$

A function $G : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is said to have **linear growth with rate** $\tilde{L}_G$ if
$\forall t \in [0,T], x \in \mathbb{R}^d\ |G(t,x)| \le \tilde{L}_G(1+|x|)$.
We remark that if G is Lipschitz with respect to x with constant L_G and $m_G := \sup_{t\in[0,T]} |G(t,0)|$ then G has linear growth with rate $\max(L_G, m_G)$. Let $(\Omega, \mathcal{F})$ be a measured space and E a Polish space. A map $\eta : (\Omega, \mathcal{F}) \longrightarrow (\mathcal{P}(E), \mathcal{B}(\mathcal{P}(E)))$ will be called **random probability** (or **random probability kernel**) if it is measurable. We will indicate by $\mathcal{P}^\Omega(E)$ the space of random probabilities. If $\mathcal{P}(E)$ is replaced by the set $\mathcal{M}(E)$ of finite non-negative measures, we will use the term **random measure** instead of **random probability**.

Remark 2.2. *Let $\eta : (\Omega, \mathcal{F}) \longrightarrow (\mathcal{M}(E), \mathcal{B}(\mathcal{M}(E)))$. η is a random measure if and only if the two following conditions hold:*

- *for each $\bar{\omega} \in \Omega$, $\eta_{\bar{\omega}} \in \mathcal{M}(E)$,*
- *for all Borel set $A \in \mathcal{B}(\mathcal{M}(E))$, $\bar{\omega} \mapsto \eta_{\bar{\omega}}(A)$ is $\mathcal{F}$-measurable.*

This was highlighted in Remark 3.20 in [10] (see also Proposition 7.25 in [7]) for the case of random probabilities. This argument can be easily adapted in the general case.

Remark 2.3. *Given $\mathbb{R}^d$-valued continuous processes $Y^1, \cdots, Y^n$, the application $\frac{1}{N}\sum_{j=1}^{N}\delta_{Y^j}$ is a random probability on $\mathcal{P}(\mathcal{C}^d)$. In fact $\delta_{Y^j}, 1 \le j \le N$ is a random probability by Remark 2.2.*

In this article will intervene some assumptions, as described below.

Assumption 1. 1. *Φ and g are Lipschitz functions defined on $[0,T]\times\mathbb{R}^d\times\mathbb{R}$ taking values respectively in $\mathbb{R}^{d\times p}$ (space of $d\times p$ matrices) and $\mathbb{R}^d$: there exist finite positive reals L_Φ and L_g such that for any $(t,y,y',z,z')\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$, we have*

$$|\Phi(t,y',z')-\Phi(t,y,z)| \le L_\Phi(|z'-z|+|y'-y|) \quad \textit{and} \quad |g(t,y',z')-g(t,y,z)| \le L_g(|z'-z|+|y'-y|)\ .$$

2. *Λ is a Borel real valued function defined on $[0,T]\times\mathbb{R}^d\times\mathbb{R}$ Lipschitz w.r.t. the space variables: there exists a finite positive real, L_Λ such that for any $(t,y,y',z,z')\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$, we have*

$$|\Lambda(t,y,z)-\Lambda(t,y',z')| \le L_\Lambda(|y'-y|+|z'-z|)\ .$$

3. *Λ is supposed to be uniformly bounded: there exist a finite positive real M_Λ such that, for any $(t,y,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}$,*

$$|\Lambda(t,y,z)| \le M_\Lambda\ .$$

4. *$K:\mathbb{R}^d\to\mathbb{R}_+$ will be a fixed regularization kernel such that $\int_{\mathbb{R}^d}K(x)dx=1$. Moreover we will suppose that it is bounded and Lipschitz: in particular we designate by M_K and L_K two positive reals such that for any $(y,y')\in\mathbb{R}^d\times\mathbb{R}^d$*

$$|K(y)| \le M_K, \quad |K(y')-K(y)| \le L_K|y'-y|\ .$$

5. *ζ_0 is a fixed Borel probability measure on $\mathbb{R}^d$ admitting a second order moment.*

6. *The functions $s\in[0,T]\mapsto\Phi(s,0,0)$ and $s\in[0,T]\mapsto g(s,0,0)$ are bounded. m_Φ (resp. m_g) will denote the quantity $\sup_{s\in[0,T]}|\Phi(s,0,0)|$ (resp. $\sup_{s\in[0,T]}|g(s,0,0)|$).*

Given a finite Borel measure γ on $\mathbb{R}^d$, $K*\gamma$ will denote the convolution function $x\mapsto\int_{\mathbb{R}^d}K(x-y)\gamma(dy)$.

To simplify we introduce the following notations.

- $V\ :[0,T]\times\mathcal{C}^d\times\mathcal{C}\to\mathbb{R}$ defined for any pair of functions $y\in\mathcal{C}^d$ and $z\in\mathcal{C}$, by

$$V_t(y,z) := \exp\left(\int_0^t \Lambda(s,y_s,z_s)ds\right) \quad \text{for any } t\in[0,T]\ . \tag{2.4}$$

- The real valued process Z such that $Z_s=u(s,Y_s)$, for any $s\in[0,T]$, will often be denoted by $u(Y)$.

With these new notations, the second equation in (1.4) can be rewritten as

$$\nu_t(\varphi) = \mathbb{E}[(\hat{K}*\varphi)(Y_t)V_t(Y,u(Y))]\ , \quad \text{for any } \varphi\in\mathcal{C}_b(\mathbb{R}^d)\ , \tag{2.5}$$

where $u(t,\cdot)=\frac{d\nu_t}{dx}$ and $\hat{K}(x)=K(-x)$.

Remark 2.4. *Under Assumption 1, item 3 (b), Λ is bounded. Consequently*

$$0 \le V_t(y,z) \le e^{tM_\Lambda} \ , \quad \text{for any } (t,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \ . \tag{2.6}$$

Under Assumption 1, item 2. Λ is Lipschitz. Then V inherits in some sense this property. Indeed, let $y,\ y' \in \mathcal{C}^d = \mathcal{C}([0,T],\mathbb{R}^d)$ and $z,\ z' \in \mathcal{C}([0,T],\mathbb{R})$. Taking $a = \int_0^t \Lambda(s,y_s,z_s)ds$ and $b = \int_0^t \Lambda(s,y'_s,z'_s)ds$ in the equality

$$e^b - e^a = (b-a)\int_0^1 e^{\alpha b + (1-\alpha)a} d\alpha \le e^{\sup(a,b)}|b-a|, \quad \forall (a,b) \in \mathbb{R}^2, \tag{2.7}$$

we obtain

$$\begin{aligned} |V_t(y',z') - V_t(y,z)| &\le e^{tM_\Lambda}\int_0^t |\Lambda(s,y'_s,z'_s) - \Lambda(s,y_s,z_s)|\, ds \\ &\le e^{tM_\Lambda} L_\Lambda \int_0^t (|y'_s - y_s| + |z'_s - z_s|)\, ds \ . \end{aligned} \tag{2.8}$$

In Section 4, Assumption 1 will be replaced by what follows.

Assumption 2. *1. All the items of Assumption 1 are in force excepted 2. which is replaced by the following.*

2. Λ is a real valued function defined on $[0,T] \times \mathbb{R}^d \times \mathbb{R}$ uniformly continuous w.r.t. the space variables (on each compact) uniformly in the time variable, see e.g. (2.3).

Remark 2.5. *The second item in Assumption 2 is fulfilled if the function Λ is continuous with respect to $(t,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$.*

In Section 5 we will treat the case when only weak solutions (in law) exist. In this case we will assume the following.

Assumption 3. *Items 3. and 4. of Assumption 1 are still in force. Besides we assume that $\Phi : [0,T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^{d\times p}$, $\quad g : [0,T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^d$ and $\Lambda : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are uniformly continuous (on each compact) with respect to the space variables uniformly in the time variable and Φ, g are uniformly bounded.*

Definition 2.6. *1. We say that* (1.4) *admits* **strong existence** *if for any filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ equipped with an $(\mathcal{F}_t)_{t\ge 0}$-Brownian motion W, an $\mathcal{F}_0$-random variable Y_0 distributed according to ζ_0, there is a couple (Y,u) where Y is an $(\mathcal{F}_t)_{t\ge 0}$-adapted process and $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, verifies* (1.4).

2. We say that (1.4) *admits* **pathwise uniqueness** *if for any filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ equipped with an $(\mathcal{F}_t)_{t\ge 0}$-Brownian motion W, an $\mathcal{F}_0$-random variable Y_0 distributed according to ζ_0, the following holds. Given two pairs (Y^1,u^1) and (Y^2,u^2) as in item 1., verifying* (1.4) *such that $Y_0^1 = Y_0^2$ $\mathbb{P}$-a.s. then $u^1 = u^2$ and Y^1 and Y^2 are indistinguishable.*

Definition 2.7. *1. We say that* (1.4) *admits* **existence in law** *(or* **weak existence***) if there is a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ equipped with an $(\mathcal{F}_t)_{t\ge 0}$-Brownian motion W, a pair (Y,u), verifying* (1.4)*, where Y is an $(\mathcal{F}_t)_{t\ge 0}$-adapted process and u is a real valued function defined on $[0,T] \times \mathbb{R}^d$.*

2. We say that (1.4) *admits* **uniqueness in law** *(or* **weak uniqueness***), if the following holds. Let $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ (resp. $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathcal{F}}_t,\tilde{\mathbb{P}})$) be a filtered probability space. Let (Y^1,u^1) (resp. $(\tilde Y^2,\tilde u^2)$) be a solution of* (1.4)*. Then $u^1 = \tilde u^2$ and Y^1 and $\tilde Y^2$ have the same law.*

3 Existence and uniqueness of the problem in the Lipschitz case

In this section we will fix a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ equipped with an $(\mathcal{F}_t)$-Brownian motion (W_t). We will proceed in two steps. We first study in Section 3.1, the second equation of (1.4) defining u. Then, in Section 3.2, we will address the equation defining the process Y.

Later in this section, Assumption 1 will be in force, in particular ζ_0 will be supposed to have a second order moment.

3.1 Existence/uniqueness and regularity of a solution to the linking equation

This subsection relies only on items 2., 3. and 4. of Assumption 1.
Here, we focus on equation (1.6) which links a probability measure m on the canonical space $\mathcal{C}^d$ into a function u defined on $[0,T]\times\mathbb{R}^d$. When $\Lambda = 0$, i.e. in the conservative case, (1.6) gives $u(t,\cdot) = K * m_t$, where m_t is the marginal law of X_t under m. Informally speaking, when K is the Delta Dirac measure, then $u(t,\cdot) = m_t$.
More precisely, for a given probability measure $m \in \mathcal{P}(\mathcal{C}^d)$, let us consider the equation

$$\begin{cases} u(t,y) = \int_{\mathcal{C}^d} K(y - X_t(\omega)) V_t(X(\omega), u(X(\omega))) dm(\omega) \ , \quad \text{for all} \quad t \in [0,T], y \in \mathbb{R}^d \ , \quad \text{with} \\ \\ V_t(X(\omega), u(X(\omega))) = \exp\left(\int_0^t \Lambda\big(s, X_s(\omega), u(s, X_s(\omega))\big) ds\right) , \end{cases} \tag{3.1}$$

where we recall that X denotes the canonical process $X : \mathcal{C}^d \to \mathcal{C}^d$ defined by $X_t(\omega) = \omega(t), \ t \geq 0, \omega \in \mathcal{C}^d$.
Equation (3.1) will be called **linking equation**: it constitutes the second line of the solution of (1.4).

The aim of this section consists in discussing existence/uniqueness and regularity of the solution of (3.1). This includes the study of the dependence with respect to m, towards two metrics on $\mathcal{P}_2(\mathcal{C}^d)$.

We first state the result about well-posedness of (3.1).

Theorem 3.1. *We assume the validity of items* 2., 3. *and* 4. *of Assumption 1.*
For a given probability measure $m \in \mathcal{P}(\mathcal{C}^d)$, equation (3.1) *admits a unique solution, u^m.*

Remark 3.2.

1. *For $(m, y) \in \mathcal{P}(\mathcal{C}^d)\times\mathbb{R}^d$, $t \mapsto u^m(t, y)$ is continuous. This follows by an application of Lebesgue dominated convergence theorem in* (3.1).

2. *Since Λ is bounded, and K is Lipschitz, it is clear that if $u := u^m$ is a solution of* (3.1) *then we have the following.*

   - $\sup |u| \leq M_K \exp(M_\Lambda T)$.
   - *u is Lipschitz with respect to the second variable with Lipschitz constant $L_K \exp(M_\Lambda T)$. Indeed, for $t \in [0,T]$*

$$\begin{aligned} |u(t,x) - u(t,y)| &\leq \int_{\mathcal{C}^d} \big|K(x - X_t(\omega)) - K(y - X_t(\omega))\big| \exp\big(\int_0^t \Lambda(r, X_r(\omega), u(r, X_r(\omega)))dr\big)] \\ &\leq L_K \exp(M_\Lambda T)|x-y|. \end{aligned} \tag{3.2}$$

Proof of Theorem 3.1. Let us introduce the linear space $\mathcal{C}_1$ of real valued continuous processes Z on $[0,T]$ (defined on the canonical space $\mathcal{C}^d$) such that

$$\|Z\|_{\infty,1} := \mathbb{E}^m \left[\sup_{t\leq T} |Z_t|\right] := \int_{\mathcal{C}^d} \sup_{0\leq t\leq T} |Z_t(\omega)| dm(\omega) < \infty \ .$$

$(\mathcal{C}_1, \|\cdot\|_{\infty,1})$ is a Banach space. For any $M \geq 0$, a well-known equivalent norm to $\|\cdot\|_{\infty,1}$ is given by $\|\cdot\|^M_{\infty,1}$, where $\|Z\|^M_{\infty,1} = \mathbb{E}^m\left[\sup_{t\leq T} e^{-Mt}|Z_t|\right]$. Let us define the operator $T^m : \mathcal{C}_1 \to \mathcal{C}([0,T]\times\mathbb{R}^d, \mathbb{R})$ such that for any $Z \in \mathcal{C}_1$,

$$T^m(Z)(t,y) := \int_{\mathcal{C}^d} K\big(y - X_t(\omega)\big)V_t\big(X(\omega), Z(\omega)\big)dm(\omega). \tag{3.3}$$

Then we introduce the operator $\tau : f \in \mathcal{C}([0,T]\times\mathbb{R}^d,\mathbb{R}) \mapsto \tau(f) \in \mathcal{C}_1$, where $\tau(f)_t(\omega) = f(t,\omega_t)$. We observe that $\tau \circ T^m$ is a map $\mathcal{C}_1 \to \mathcal{C}_1$.

Notice that equation (3.1) is equivalent to

$$u = (T^m \circ \tau)(u). \tag{3.4}$$

We first admit the existence and uniqueness of a fixed point $Z \in \mathcal{C}_1$ for the map $\tau \circ T^m$. In particular we have $Z = (\tau \circ T^m)(Z)$. We can now deduce the existence/uniqueness for the function u for problem (3.4). Concerning existence, we choose $v^m := T^m(Z)$. Since Z is a fixed-point of the map $\tau \circ T^m$, by the definition of v^m we have

$$Z = \tau(T^m(Z)), \tag{3.5}$$

so that v^m is a solution of (3.4).
Concerning uniqueness of (3.4), we consider two solutions of (3.4) $\bar{v}, \tilde{v}$, i.e. such that $\bar{v} = (T^m \circ \tau)(\bar{v}), \tilde{v} = (T^m \circ \tau)(\tilde{v})$. We set $\bar{Z} := \tau(\bar{v}), \tilde{Z} := \tau(\tilde{v})$. Since $\bar{v} = T^m(\bar{Z})$ we have $\bar{Z} = \tau(\bar{v}) = \tau(T^m(\bar{Z}))$. Similarly $\tilde{Z} = \tau(\tilde{v}) = \tau(T^m(\tilde{Z}))$. Since $\bar{Z}$ and $\tilde{Z}$ are fixed points of $\tau \circ T^m$, it follows that $\bar{Z} = \tilde{Z}$ dm a.e. Finally $\bar{v} = T^m(\bar{Z}) = T^m(\tilde{Z}) = \tilde{v}$. It remains finally to prove that $\tau \circ T^m$ admits a unique fixed point, Z.
The upper bound (2.8) implies that for any pair $(Z, Z') \in \mathcal{C}_1 \times \mathcal{C}_1$, for any $(t,y) \in [0,T]\times\mathbb{R}^d$,

$$\begin{aligned}
|T^m(Z') - T^m(Z)|(t,y) &= \left|\int_{\mathcal{C}^d} K(y - X_t(\omega))\left[V_t(X(\omega), Z'(\omega)) - V_t(X(\omega), Z(\omega))\right]dm(\omega)\right| \\
&\leq M_K e^{tM_\Lambda} L_\Lambda \int_{\mathcal{C}^d}\int_0^t |Z'_s(\omega) - Z_s(\omega)|ds\, dm(\omega) \\
&\leq M_K e^{TM_\Lambda} L_\Lambda \mathbb{E}\left[\int_0^t e^{Ms}e^{-Ms}|Z'_s - Z_s|ds\right] \\
&\leq M_K e^{TM_\Lambda} L_\Lambda \mathbb{E}\left[\int_0^t e^{Ms}\sup_{r\leq t} e^{-Mr}|Z'_r - Z_r|ds\right] \\
&\leq M_K e^{TM_\Lambda} L_\Lambda \frac{e^{Mt}-1}{M}\mathbb{E}\left[\sup_{r\leq t} e^{-Mr}|Z'_r - Z_r|\right] \\
&\leq M_K e^{TM_\Lambda} L_\Lambda \frac{e^{Mt}-1}{M}\|Z' - Z\|^M_{\infty,1}\,.
\end{aligned}$$

Then considering $(\tau \circ T^m)(Z')_t = T^m((Z')(t,X_t)$ and $(\tau \circ T^m)(Z)_t = T(Z)(t,X_t)$, we obtain

$$\begin{aligned}
\sup_{t\leq T} e^{-Mt}\left|(\tau \circ T^m)(Z')_t - (\tau \circ T^m)(Z)_t\right| &= \sup_{t\leq T} e^{-Mt}\left|T^m(Z')(t,X_t) - T^m(Z)(t,X_t)\right| \\
&\leq M_K e^{TM_\Lambda} L_\Lambda \frac{1}{M}\|Z' - Z\|^M_{\infty,1}\,.
\end{aligned}$$

Taking the expectation yields $|(\tau \circ T^m)(Z')_t - (\tau \circ T^m)(Z)_t\|^M_{\infty,1} \leq M_K e^{TM_\Lambda} L_\Lambda \frac{1}{M}\|Z' - Z\|^M_{\infty,1}$. Hence, as soon as M is sufficiently large, $M > M_K e^{TM_\Lambda} L_\Lambda$, $(\tau \circ T^m)$ is a contraction on $(\mathcal{C}_1, \|\cdot\|^M_{\infty,1})$ and the proof ends by a simple application of the Banach fixed point theorem. $\square$

In the sequel, we will need a stability result on u^m solution of (3.1), w.r.t. the probability measure m, which will be treated in the fundamental proposition below. The proof will be postponed in the Appendix, see Section 7.1.

Proposition 3.3. *We assume the validity of items 2., 3. and 4. of Assumption 1.*
Let u be a solution of (3.1). The following assertions hold.

1. *For any couple of probabilities $(m,m') \in \mathcal{P}_2(\mathcal{C}^d) \times \mathcal{P}_2(\mathcal{C}^d)$, for all $(t,y,y') \in [0,T] \times \mathcal{C}^d \times \mathcal{C}^d$, we have*
$$|u^m(t,y) - u^{m'}(t,y')|^2 \leq C_{K,\Lambda}(t)\left[|y-y'|^2 + |\mathcal{W}_t(m,m')|^2\right] \ , \tag{3.6}$$
where $C_{K,\Lambda}(t) := 2C'_{K,\Lambda}(t)(t+2)(1+e^{2tC'_{K,\Lambda}(t)})$ with $C'_{K,\Lambda}(t) = 2e^{2tM_\Lambda}(L_K^2 + 2M_K^2L_\Lambda^2 t)$. In particular the functions $C_{K,\Lambda}$ only depend on $M_K, L_K, M_\Lambda, L_\Lambda$ and t and are increasing with t.

2. *For any $(m,m') \in \mathcal{P}(C^d) \times \mathcal{P}(C^d)$, for all $(t,y,y') \in [0,T] \times \mathcal{C}^d \times \mathcal{C}^d$, we have*
$$|u^m(t,y) - u^{m'}(t,y')|^2 \leq \mathfrak{C}_{K,\Lambda}(t)\left[|y-y'|^2 + |\widetilde{\mathcal{W}}_t(m,m')|^2\right] \ , \tag{3.7}$$
where $\mathfrak{C}_{K,\Lambda}(t) := 2\mathfrak{C}'_{K,\Lambda}(t)(t+2)(1+e^{2t\mathfrak{C}'_{K,\Lambda}(t)})$ with $\mathfrak{C}'_{K,\Lambda}(t) := 2e^{2tM_\Lambda}(\max(L_K, 2M_K)^2 + 2M_K^2 \max(L_\Lambda, 2M_\Lambda)^2 t)$.

3. *The map $(m,t,x) \mapsto u^m(t,x)$ is continuous on $\mathcal{P}(\mathcal{C}^d) \times [0,T] \times \mathbb{R}^d$ where $\mathcal{P}(\mathcal{C}^d)$ is endowed with the topology of weak convergence.*

4. *Suppose that $K \in W^{1,2}(\mathbb{R}^d)$. Then for any $(m,m') \in \mathcal{P}_2(\mathcal{C}^d) \times \mathcal{P}_2(\mathcal{C}^d)$, $t \in [0,T]$*
$$\|u^m(t,\cdot) - u^{m'}(t,\cdot)\|_2^2 \leq \tilde{C}_{K,\Lambda}(t)(1+2tC_{K,\Lambda}(t))|\mathcal{W}_t(m,m')|^2 \ , \tag{3.8}$$
where $C_{K,\Lambda}(t) := 2C'_{K,\Lambda}(t)(t+2)(1+e^{2tC'_{K,\Lambda}(t)})$ with $C'_{K,\Lambda}(t) = 2e^{2tM_\Lambda}(L_K^2 + 2M_K^2L_\Lambda^2 t)$ and $\tilde{C}_{K,\Lambda}(t) := 2e^{2tM_\Lambda}(2M_KL_\Lambda^2 t(t+1) + \|\nabla K\|_2^2)$, recalling that $\|\cdot\|_2$ denotes the standard $L^2(\mathbb{R}^d)$ or $L^2(\mathbb{R}^d,\mathbb{R}^d)$-norms. Moreover $t \mapsto \tilde{C}_{K,\Lambda}, C_{K,\Lambda}$ are increasing in t.

5. *Suppose that $\mathcal{F}(K) \in L^1(\mathbb{R}^d)$. Then there exists a constant $\bar{C}_{K,\Lambda}(t) > 0$ (depending only on $t, M_\Lambda, L_\Lambda, \|\mathcal{F}(K)\|_1$) such that for any random probability $\eta : (\Omega,\mathcal{F}) \longrightarrow (\mathcal{P}(\mathcal{C}^d), \mathcal{B}(\mathcal{P}(\mathcal{C}^d)))$, for all $(t,m) \in [0,T] \times \mathcal{P}(\mathcal{C}^d)$*
$$\mathbb{E}[\|u^\eta(t,\cdot) - u^m(t,\cdot)\|_\infty^2] \quad \leq \quad \bar{C}_{K,\Lambda}(t) \sup_{\substack{\varphi \in \mathcal{C}_b(\mathcal{C}^d) \\ \|\varphi\|_\infty \leq 1}} \mathbb{E}[|\langle \eta - m, \varphi\rangle|^2]. \tag{3.9}$$
We remark that the expectation in both sides of (3.9) is taken w.r.t. the randomness of the random probability η.

Remark 3.4. *a) By Corollary 6.13, Chapter 6 in [21], $\widetilde{\mathcal{W}}_T$ is a metric compatible with the weak convergence on $\mathcal{P}(\mathcal{C}^d)$.*

b) The map $d_2^\Omega : (\nu,\mu) \mapsto \sqrt{\sup_{\substack{\varphi \in \mathcal{C}_b(\mathcal{C}^d) \\ \|\varphi\|_\infty \leq 1}} \mathbb{E}[|\langle \nu - \mu, \varphi\rangle|^2]}$ defines a (homogeneous) distance on $\mathcal{P}^\Omega(\mathcal{C}^d)$. That distance is used in [13] in order to control the error induced by the interacting particle approximation scheme.

c) The Lipschitz continuity stated in item 2. of Proposition 3.3 implies the one of item 1. For expository reasons, we have decided to start with the less general case.

To conclude this part, we want to highlight some properties of the function u^m, which will is used in Section 3 of [13]. In fact, the map $(m,t,x) \in \mathcal{P}(\mathcal{C}^d) \times [0,T] \times \mathbb{R}^d \mapsto u^m(t,x)$ has an important non-anticipating property. We begin by defining the notion of induced measure. For the rest of this section, we fix $t \in [0,T]$.

Definition 3.5. *Given a non-negative Borel measure m on $(\mathcal{C}^d, \mathcal{B}(\mathcal{C}^d))$. From now on, m_t will denote the (unique) induced measure on $(\mathcal{C}_t^d, \mathcal{B}(\mathcal{C}_t^d))$ (with $\mathcal{C}_t^d := \mathcal{C}([0,t],\mathbb{R}^d)$) defined by*
$$\int_{\mathcal{C}_t^d} F(\phi) m_t(d\phi) = \int_{\mathcal{C}^d} F(\phi_{|[0,t]}) m(d\phi),$$
where $F : \mathcal{C}_t^d \longrightarrow \mathbb{R}$ is bounded and continuous.

Remark 3.6. *Let $t \in [0,T], m = \delta_\xi \ , \xi \in \mathcal{C}^d$. The induced measure, m_t, on $\mathcal{C}^d_t$ is $\delta_{(\xi_r | 0 \le r \le t)}$.*

The same construction as the one carried on in Theorem 3.1 allows us to define the unique solution to

$$u^{m_t}(s,y) = \int_{\mathcal{C}^d_t} K(y - X_s(\omega)) \exp\left(\int_0^s \Lambda(r, X_r(\omega), u^{m_t}(r, X_r(\omega)))dr\right) m_t(d\omega) \quad \forall s \in [0,t] \ . \tag{3.10}$$

Proposition 3.7. *Under the assumption of Theorem 3.1, we have*

$$\forall (s,y) \in [0,t] \times \mathbb{R}^d, \ u^m(s,y) = u^{m_t}(s,y).$$

Proof. By definition of m_t, it follows that $(s,y) \mapsto u^m(s,y)|_{[0,t]\times\mathbb{R}^d}$ is a solution of (3.10). Invoking the uniqueness of (3.10) ends the proof. □

Corollary 3.8. *Let $N \in \mathbb{N}$, $\xi^1, \cdots, \xi^i, \cdots, \xi^N$ be $(\mathcal{G}_t)$-adapted continuous processes, where $\mathcal{G}$ is a filtration (defined on some probability space) fulfilling the usual conditions. Let $m(d\omega) = \frac{1}{N}\sum_{i=1}^N \delta_{\xi^i}(d\omega)$. Then, $(u^m(t,y))$ is a $(\mathcal{G}_t)$-adapted random field, i.e. for any $(t,y) \in [0,T] \times \mathbb{R}^d$, the process is $(\mathcal{G}_t)$-adapted.*

3.2 Existence and uniqueness of the solution to the McKean stochastic differential equations

For a given $m \in \mathcal{P}_2(\mathcal{C}^d)$, u^m is well-defined according to Theorem 3.1. Let $Y_0 \sim \zeta_0$. The well-posedness of (1.4) is equivalent to the one related to the following McKean type SDE:

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s, u^m(s, Y_s))dW_s + \int_0^t g(s, Y_s, u^m(s, Y_s))ds \\ m = \mathcal{L}(Y) \ . \end{cases} \tag{3.11}$$

The aim of the present section is to prove, following Sznitman [18], by a fixed point argument the following result.

Theorem 3.9. *Under Assumption 1, the McKean type SDE* (1.4) *admits the following properties.*

1. *Strong existence and pathwise uniqueness;*

2. *existence and uniqueness in law.*

Proof of Theorem 3.9. We fix $m \in \mathcal{P}_2(\mathcal{C}^d)$. Thanks to Assumption 1 and Proposition 3.3 implying the Lipschitz property of u^m w.r.t. the space variable (uniformly in time), the first line of (3.11) admits a unique strong solution Y^m, for which by classical arguments as Burkholder-Davies-Gundy (BDG) and Jensen's inequality, there exists a positive real $C_0 = C_0(L_\Phi, L_g, m_\Phi, m_g) > 0$ such that $\mathbb{E}[\sup_{t \le T} |Y_t|^2] \le C_0\left(1 + \mathbb{E}[|Y_0|^2]\right)$. Consequently the law $\Theta(m) := \mathcal{L}(Y^m)$ belongs to $\mathcal{P}_2(\mathcal{C}^d)$. We consider now the application $\Theta : \mathcal{P}_2(\mathcal{C}^d) \to \mathcal{P}_2(\mathcal{C}^d)$.

Let now m and m' in $\mathcal{P}_2(\mathcal{C}^d)$. We are interested in proving that Θ is a contraction for the Wasserstein metric. Let $u := u^m, u' := u^{m'}$ be solutions of (3.1) related to m and m'. Let Y (resp. Y') be the solution of the first line of (3.11) related to m (resp. m').
By definition of the Wasserstein metric (2.1)

$$|\mathcal{W}_T(\Theta(m), \Theta(m'))|^2 \le \mathbb{E}[\sup_{t \le T} |Y'_t - Y_t|^2] \ . \tag{3.12}$$

Hence, we control $|Y'_t - Y_t|$ with the help of Lemma 7.1 in the Appendix.

Using the overmentioned Lemma 7.1 and Proposition 3.3, by applying successively inequalities (7.33) and (3.6), gives

$$\mathbb{E}[\sup_{t\leq a}|Y'_t - Y_t|^2] \leq C\left[\int_0^a \mathbb{E}[\sup_{s\leq t}|Y'_s - Y_s|^2]dt + \int_0^a |\mathcal{W}_t(m,m')|^2 dt\right] , \tag{3.13}$$

for any $a \in [0,T]$, where $C = C_{\Phi,g}(T)C_{K,\Lambda}(T)$.
Applying Gronwall's lemma to (3.13) yields

$$\mathbb{E}[\sup_{t\leq a}|Y_t - Y'_t|^2] \leq Ce^{CT}\int_0^a |\mathcal{W}_s(m,m')|^2 ds \, . \tag{3.14}$$

Then recalling (3.12), this finally gives

$$|\mathcal{W}_a(\Theta(m),\Theta(m'))|^2 \leq Ce^{CT}\int_0^a |\mathcal{W}_s(m,m')|^2 ds, \; a\in[0,T]. \tag{3.15}$$

We end the proof of item 1. by classical fixed point argument, similarly to the one of Chapter 1, section 1 of Sznitman [18].
Concerning item 2. it remains to show uniqueness in law for (1.4). Let $(Y^1,m^1),(Y^2,m^2)$ be two solutions of (3.11) on different probability spaces, Brownian motions and initial conditions distributed according to ζ_0. Given $m\in\mathcal{P}_2(\mathcal{C}^d)$, we indicate by $\Theta(m)$ the law of $\bar{Y}$, where $\bar{Y}$ is the (strong) solution of

$$\bar{Y}_t = Y_0^1 + \int_0^t \Phi(s,\bar{Y}_s,u^{m^2}(s,\bar{Y}_s))dW_s + \int_0^t g(s,\bar{Y}_s,u^{m^2}(s,\bar{Y}_s))ds \, , \tag{3.16}$$

on the same probability space and same Brownian motion on which Y^1 lives. Since u^{m^2} is fixed, $\bar{Y}$ is solution of a classical SDE with Lipschitz coefficients for which pathwise uniqueness holds. By Yamada-Watanabe theorem, Y^2 and $\bar{Y}$ have the same distribution. Consequently, $\Theta(m^2)=\mathcal{L}(\bar{Y})=\mathcal{L}(Y^2)=m^2$. It remains to show that $Y^1=\bar{Y}$ in law, i.e. $m^1=m^2$. By the same arguments as for the proof of 1., we get (3.15), i.e. for all $a\in[0,T]$,

$$|\mathcal{W}_a(\mathcal{L}(Y^1),\mathcal{L}(\bar{Y}))|^2 = |\mathcal{W}_a(\Theta(m^1),\Theta(m^2))|^2 \leq Ce^{CT}\int_0^a |\mathcal{W}_s(m^1,m^2)|^2 ds.$$

Since $\Theta(m^1)=m^1$ and $\Theta(m^2)=m^2$, by Gronwall's lemma $m^1=m^2$ and finally $Y^1=\bar{Y}$ (in law). This concludes the proof of Proposition 3.9. □

4 Strong Existence under weaker assumptions

Let us fix a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ equipped with a p dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $(W_t)_{t\geq 0}$.
In this section Assumption 2 will be in force. In particular, we suppose that ζ_0 is a Borel probability measure having a second order moment.

Before proving the main result of this part, we remark that in this case, uniqueness fails for (1.4). To illustrate this, we consider the following counterexample, which is even valid for $d=1$.

Example 4.1. *Consider the case $\Phi=g=0$, $Y_0=0$ so that $\zeta_0=\delta_0$. This implies that $Y_t\equiv 0$ is a strong solution of the first line of* (1.4). *Since $u(0,.)=(K*\zeta_0)(\cdot)$, we have $u(0,\cdot)=K$.*
A solution u of the second line equation of (1.4), *will be of the form*

$$u(t,y) = K(y)\exp\left(\int_0^t \Lambda(r,0,u(r,0))dr\right), \tag{4.1}$$

for some suitable Λ fulfilling Assumption 2, item 2. We will in fact consider Λ independent of the time and $\beta(u) := \Lambda(0,0,u)$. Without restriction of generality we can suppose $K(0) = 1$. We will show that the second line equation of (1.4) *is not well-posed for some particular choice of β. Now* (4.1) *becomes*

$$u(t,y) = K(y)\exp\left(\int_0^t \beta(u(r,0))dr\right). \tag{4.2}$$

By setting $y = 0$, we get $\phi(t) := u(t,0)$ and in particular, necessarily we have

$$\phi(t) = \exp\left(\int_0^t \beta(\phi(r))dr\right). \tag{4.3}$$

A solution u given in (4.2) *is determined by setting $u(t,y) = K(y)\phi(t)$. Now, we choose the function β such that for given constants $\alpha \in (0,1)$ and $C > 1$,*

$$\beta(r) = \begin{cases} |r-1|^\alpha & , \text{ if } r \in [0,C] \\ |C-1|^\alpha & , \text{ if } r \geq C \\ 1 & , \text{ if } r \leq 0\,. \end{cases} \tag{4.4}$$

β is clearly a bounded, uniformly continuous function verifying $\beta(1) = 0$ and $\beta(r) \neq 0$, for all $r \neq 1$.

We define $F : \mathbb{R} \longrightarrow \mathbb{R}$, by $F(u) = \int_1^u \frac{1}{r\beta(r)}dr$. F is strictly positive on $(1,+\infty)$, and it is a homeomorphism from $[1,+\infty)$ to $\mathbb{R}_+$, since $\int_1^{+\infty} \frac{1}{r\beta(r)}dr = \infty$.

On one hand, by setting $\phi(t) := F^{-1}(t)$, for $t > 0$, we observe that ϕ verifies $\phi'(t) = \phi(t)\beta(\phi(t)), t > 0$ and so ϕ is a solution of (4.3). *On the other hand, the function $\phi \equiv 1$ also satisfies* (4.3), *with the same choice of Λ, related to β. This shows the non-uniqueness for the second equation of* (1.4).

The main theorem of this section states existence (even though non-uniqueness) for (1.4), when only the coefficients Φ and g of the SDE are Lipschitz in (x,u). The idea of this section is to regularize the coefficient Λ into $\Lambda_n := \Lambda * \phi_n^{d+1}$, to make use of results of Section 3 and then to control the limit.

Theorem 4.2. *Under Assumption 2,* (1.4) *admits strong existence.*

The proof uses three main ingredients.

1. The tightness of processes $(Y^n)_{n\in\mathbb{N}}$ corresponding to the solutions $(Y_n, u_n)_{n\in\mathbb{N}}$ related to Λ_n, see Lemma 7.10 in the Appendix.

2. Given a sequence $(Y_n)_{n\in\mathbb{N}}$ of processes converging in law to some Borel probability measure m on $\mathcal{C}^d$, we show the convergence of $(u_n)_{n\in\mathbb{N}}$ to some function u verifying the linking equation (3.1) related to m, see Proposition 4.3.

3. The strong convergence of $(Y_n)_{n\in\mathbb{N}}$ to some process Y, whose law is m, see Lemma 7.8 in the Appendix.

Before proving the main result, we first establish proposition below, permitting us to prove the statement 2. above.

Proposition 4.3. *Let $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of Borel uniformly bounded functions defined on $[0,T]\times\mathbb{R}^d\times\mathbb{R}$ such that for every $n \in \mathbb{N}$, $\Lambda_n(t,\cdot,\cdot)$ is continuous. Assume the existence of a Borel function $\Lambda : [0,T]\times\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$ such that for almost all $t \in [0,T]$, $[\Lambda_n(t,\cdot,\cdot) - \Lambda(t,\cdot,\cdot)] \xrightarrow[n\to+\infty]{} 0$ uniformly on each compact of $\mathbb{R}^d\times\mathbb{R}$. Let (Y^n)*

be a sequence of $\mathbb{R}^d$-valued continuous processes, whose law is denoted by m^n.
We set $Z^n := u_n(.\,,Y^n)$, where for any $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$u_n(t,x) = \int_{\mathcal{C}^d} K(x - X_t(\omega)) \, \exp\left\{\int_0^t \Lambda_n\big(r, X_r(\omega), u_n(r, X_r(\omega))\big)dr\right\} dm^n(\omega). \tag{4.5}$$

We suppose that (Y^n, Z^n) converges in law.
Then (u_n) converges uniformly on each compact to some continuous $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ which fulfills

$$u(t,x) = \int_{\mathcal{C}^d} K(x - X_t(\omega)) \exp\Big(\int_0^t \Lambda(r, X_r(\omega), u(r, X_r(\omega)))dr\Big) dm(\omega), \tag{4.6}$$

where m is the limit of $(m^n)_{n\geq 0}$.

Proof. We draw the reader's attention on the fact that all the technical results invoked in this proof are stated and proved in Subsection 7.3 of the Appendix.
Let ν denote the Borel probability measure on $\mathcal{C}^d \times \mathcal{C}$ to which the law of (Y^n, Z^n) converge. Without loss of generality, the proof below is written with $d = 1$. By Proposition 7.6, the left-hand side of (4.5) converges uniformly on each compact to the continuous function u defined by

$$u(t,x) = \int_{\mathcal{C}^d \times \mathcal{C}} K(x - X_t(\omega)) \, \exp\left\{\int_0^t \Lambda\big(r, X_r(\omega), X'_r(\omega')\big)dr\right\} d\nu(\omega, \omega'), \quad (t,x) \in [0,T] \times \mathbb{R}\ . \tag{4.7}$$

It remains to show that u fulfills (4.6). For this we will take the limit of the right-hand side (r.h.s.) of (4.5) and we will show that it gives the r.h.s. of (4.6). For $n \in \mathbb{N}$, $(r,x) \in [0,T] \times \mathbb{R}$, we set

$$\tilde{\Lambda}_n(r,x) \quad := \quad \Lambda_n(r, x, u_n(r,x)) \tag{4.8}$$
$$\tilde{\Lambda}(r,x) \quad := \quad \Lambda(r, x, u(r,x)). \tag{4.9}$$

We fix $(t,x) \in [0,T] \times \mathbb{R}$. In view of applying Lemma 7.2, we define $f_n, f : \mathcal{C} \to \mathbb{R}$ such that

$$\begin{aligned} f_n(y) &= K(x - y_t) \exp\Big(\int_0^t \tilde{\Lambda}_n(r, y_r)dr\Big) \\ f(y) &= K(x - y_t) \exp\Big(\int_0^t \tilde{\Lambda}(r, y_r)dr\Big)\ . \end{aligned}$$

We also set $\mathbb{P}^n := m^n$. Since (Y^n, Z^n) converges in law to ν, m^n converges weakly to m. Since the sequence of functions $|\tilde{\Lambda}_n|$ is uniformly bounded then the sequence of functions (f_n) is also uniformly bounded. We denote by M_Λ the common upper bound of the each Λ_n.

The maps (f_n) are continuous by Lemma 7.5, and also the function f since, u is continuous on $[0,T] \times \mathbb{R}$. Taking into account Remark 7.3, we will show that $f_n \to f$ uniformly on each ball of $\mathcal{C}$.

Let us fix $M > 0$ and set $B_1(0,M) := \{y \in \mathcal{C}, ||y||_\infty := \sup_{s\in[0,T]} |y_s| \leq M\}$. For any locally bounded function $\ell : [0,T] \times \mathbb{R} \to \mathbb{R}$, we set $||\ell||_{\infty,M} := \sup_{s\in[0,T],\xi\in[-M,M]} |\ell(s,\xi)|$. Let $\varepsilon > 0$.
Since $u_n \to u$ uniformly on $[0,T] \times [-M,M]$, there exists $n_0 \in \mathbb{N}$ such that,

$$n \geq n_0 \Longrightarrow ||u_n - u||_{\infty,M} < \varepsilon\ . \tag{4.10}$$

The sequence $u_n|_{[0,T]\times[-M,M]}$ is uniformly bounded. Let I_M be a compact interval including the subset $\{u_n(s,x) \mid (s,x) \in [0,T] \times [-M,M]\}$.

For all $(s,x) \in [0,T] \times [-M,M]$,

$$\begin{array}{rcl} |\tilde{\Lambda}_n(s,x) - \tilde{\Lambda}(s,x)| & = & |\Lambda_n(s,x,u_n(s,x)) - \Lambda(s,x,u(s,x))| \\ & \leq & |\Lambda_n(s,x,u_n(s,x)) - \Lambda(s,x,u_n(s,x))| + |\Lambda(s,x,u_n(s,x)) - \Lambda(s,x,u(s,x))| \\ & := & I_1(n,s,x) + I_2(n,s,x) \ . \end{array} \tag{4.11}$$

Concerning I_1, since for almost all $s \in [0,T]$, $\Lambda_n(s,\cdot,\cdot) \xrightarrow[n \to \infty]{} \Lambda(s,\cdot,\cdot)$ uniformly on $[-M,M] \times I_M$, we have for $x \in [-M,M]$,

$$0 \leq I_1(n,s,x) \leq \sup_{x \in [-M,M], \xi \in I_M} |\Lambda_n(s,x,\xi) - \Lambda(s,x,\xi)| \xrightarrow[n \to \infty]{} 0 \ ds\text{-a.e.} \ ,$$

from which we deduce

$$\sup_{x \in [-M,M]} I_1(n,s,x) \xrightarrow[n \to \infty]{} 0 \ ds\text{-a.e.} \tag{4.12}$$

Now, we treat the term I_2. Taking into account (4.10), we get for $n \geq n_0$ (n_0 depending on ε),

$$0 \leq \sup_{s \in [0,T], x \in [-M,M]} I_2(n,s,x) \leq S(\varepsilon), \tag{4.13}$$

where

$$S(\varepsilon) := \sup_{s \in [0,T], x \in [-M,M], |\xi_1 - \xi_2| \leq \varepsilon} |\Lambda(s,x,\xi_1) - \Lambda(s,x,\xi_2)|.$$

We take the $\limsup$ on both sides of (4.13), which gives,

$$\limsup_{n \longrightarrow \infty} \sup_{s \in [0,T], x \in [-M,M]} I_2(n,s,x) \leq S(\varepsilon). \tag{4.14}$$

Summing up (4.12), (4.14) and taking into account (4.11), we get,

$$0 \leq \limsup_{n \to \infty} \sup_{x \in [-M,M]} |\tilde{\Lambda}_n(s,x) - \tilde{\Lambda}(s,x)| \leq S(\varepsilon) \ ds\text{-a.e.} \tag{4.15}$$

Since Λ satisfies Assumption 2, the uniform continuity on each compact of $(x,\xi) \in \mathbb{R} \times \mathbb{R} \mapsto \Lambda(s,x,\xi)$ (uniformly with respect to s) holds and $\lim_{\varepsilon \longrightarrow 0} S(\varepsilon) = 0$. Finally,

$$\sup_{x \in [-M,M]} |\tilde{\Lambda}_n(s,x) - \tilde{\Lambda}(s,x)| \xrightarrow[n \to \infty]{} 0 \ ds\text{-a.e.} \tag{4.16}$$

Now, for $n \geq n_0$, using (2.7), we obtain

$$\sup_{y \in B_1(0,M)} |f_n(y) - f(y)| \quad \leq \quad M_K \exp(M_\Lambda T) \int_0^T \sup_{x \in [-M,M]} |\tilde{\Lambda}_n(r,x) - \tilde{\Lambda}(r,x)| dr. \tag{4.17}$$

Since $(\tilde{\Lambda}_n), \Lambda$ are uniformly bounded, taking into account (4.16) and Lebesgue's dominated convergence theorem, the right-hand side of (4.17) goes to 0 when $n \longrightarrow 0$. This shows that $f_n \longrightarrow f$ uniformly on $B_1(0,M)$.

We can now apply Lemma 7.2 (with $\mathbb{P}_n$ and f_n defined above) to obtain, for $n \longrightarrow \infty$,

$$\int_{\mathcal{C}} K(x - X_t(\omega)) \exp\left(\int_0^t \Lambda_n(r, X_r(\omega), u_n(r, X_r(\omega))) dr\right) dm^n(\omega)$$

converges to

$$\int_{\mathcal{C}} K(x - X_t(\omega)) \exp\left(\int_0^t \Lambda(r, X_r(\omega), u(r, X_r(\omega))) dr\right) dm(\omega),$$

which finally proves (4.6) and concludes the proof of Proposition 4.3. $\square$

Now, we are able to prove the main result of this section.

Proof of Theorem 4.2. Let Y_0 be a r.v. distributed according to ζ_0. We define

$$\Lambda_n : (t,x,\xi) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \mapsto \Lambda_n(t,x,\xi) := \int_{\mathbb{R}^d \times \mathbb{R}} \phi_n^d(x-x_1)\phi_n(\xi-\xi_1)\Lambda(t,x_1,\xi_1)dx_1 d\xi_1, \tag{4.18}$$

where $(\phi_n)_{n\geq 0}$ is a usual mollifier sequence converging (weakly) to the Dirac measure. Thanks to the classical properties of the convolution, we know that Λ being bounded implies that $\forall n \in \mathbb{N}, ||\Lambda_n||_\infty \leq ||\phi_n^d||_{L^1} ||\phi_n||_{L^1} ||\Lambda||_\infty = ||\Lambda||_\infty$. For fixed $n \in \mathbb{N}$, ϕ_n are Lipschitz so that Λ_n defined in (4.18) is also Lipschitz (and uniformly bounded). Then, for fixed $n \in \mathbb{N}$, Φ, g are Lipschitz and they have linear growth by item 6. of Assumption 1. So we can apply the results of Section 3 (see Theorem 3.9) to obtain the existence of a pair (Y^n, u_n) such that

$$\begin{cases} dY_t^n = \Phi(t, Y_t^n, u_n(t, Y_t^n))dW_t + g(t, Y_t^n, u_n(t, Y_t^n))dt \\ Y_0^n = Y_0, \\ u_n(t,x) = \mathbb{E}[K(x - Y_t^n) \exp\big(\int_0^t \Lambda_n(r, Y_r^n, u_n(r, Y_r^n))dr\big)]. \end{cases} \tag{4.19}$$

We recall that Λ_n are uniformly bounded and we remark that Φ, g have linear growth, taking into account the fact that they are Lipschitz and fulfill item 6. of Assumption 1; moreover $\{Y_0^n\}_{n\in\mathbb{N}}$ are obviously tight. Consequently Lemma 7.10 in the Appendix gives the existence of a subsequence (n_k) such that $(Y^{n_k}, u_{n_k}(\cdot, Y_\cdot^{n_k}))$ converges in law to some Borel probability measure ν on $\mathcal{C}^d \times \mathcal{C}$. By Assumption 2, for all $t \in [0,T]$, $\Lambda_n(t,\cdot,\cdot)$ converges to $\Lambda(t,\cdot,\cdot)$, uniformly on every compact subset of $\mathbb{R}^d \times \mathbb{R}$.

In view of applying Proposition 4.3, we set $Z_t^{n_k} := u_{n_k}(t, Y_t^{n_k})$ and $m^{n_k} := \mathcal{L}(Y^{n_k})$. We know that $(\Lambda_{n_k}), \Lambda$ satisfy the hypotheses of Proposition 4.3. On the other hand (Y^{n_k}, Z^{n_k}) converges in law to ν. Now Proposition 4.3 says that (u_{n_k}) converges uniformly on each compact to some u which verifies (4.6), where m is the first marginal of ν. In particular we emphasize that the sequence (Y^{n_k}) converges in law to m.

We continue the proof of Theorem 4.2 concentrating on the first line of (1.4).

We set, for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $k \in \mathbb{N}$,

$$\begin{array}{lcl} a_k(t,x) & := & \Phi(t,x,u_{n_k}(t,x)) \\ b_k(t,x) & := & g(t,x,u_{n_k}(t,x)) \\ a(t,x) & := & \Phi(t,x,u(t,x)) \\ b(t,x) & := & g(t,x,u(t,x)) \,. \end{array} \tag{4.20}$$

Here, the functions u_n being fixed, the first equation of (4.19) is a classical SDE, whose coefficients depend on the (deterministic) continuous function u_n. By item 2. of Remark 3.2, the functions u_n appearing in (4.19) are Lipschitz with respect to the second argument with constant not depending on n and uniformly bounded. This implies that the coefficients a_k, b_k are Lipschitz (with constant not depending on k) and have linear growth with uniform rate.

Since (u_{n_k}) converges pointwise to u, then (a_k), (b_k) converges pointwise respectively to a, b where $a(t,x) = \Phi(t,x,u(t,x)), b(t,x) = g(t,x,u(t,x))$.

Consequently, we can apply Lemma 7.8 with the sequence of classical SDEs

$$\begin{cases} dY_t^{n_k} = a_k(t, Y_t^{n_k})dW_t + b_k(t, Y_t^{n_k})dt \\ Y_0^{n_k} = Y_0, \end{cases} \tag{4.21}$$

to obtain

$$\sup_{t\leq T}|Y_t^{n_k}-Y_t| \xrightarrow[k \longrightarrow +\infty]{L^2(\Omega)} 0,$$

where Y is the (strong) solution to the classical SDE

$$\begin{cases} dZ_t = a(t,Z_t)dW_t + b(t,Z_t)dt \\ Z_0 = Y_0 \\ a(t,x) = \Phi(t,x,u(t,x)) \\ b(t,x) = g(t,x,u(t,x)) \ . \end{cases} \tag{4.22}$$

We remark that Y verifies the first equation of (1.4) and the corresponding u fulfills (4.6). To conclude the proof of Theorem 4.2 it remains to identify the law of Y with m. Since Y^{n_k} converges strongly, then the laws m^{n_k} of Y^{n_k} converge to the law of Y, which by Proposition 4.3, coincides necessarily to m.

□

5 Weak Existence when the coefficients are continuous

In this section we consider again (1.4) i.e. problem

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(r,Y_r,u(r,Y_r))dW_r + \int_0^t g(r,Y_r,u(r,Y_r))dr \ , \quad \text{with} \quad Y_0 \sim \zeta_0 \ , \\ u(t,x) = \int_{\mathcal{C}^d} dm(\omega) \left[K(x-X_t(\omega)) \exp\left\{\int_0^t \Lambda\big(r,X_r(\omega),u(r,X_r(\omega))\big)dr\right\}\right] \ , \quad \text{for} \quad (t,x)\in[0,T]\times\mathbb{R}^d \\ m = \mathcal{L}(Y) \ , \end{cases} \tag{5.23}$$

but without the Lipschitz conditions on the coefficients Φ, g, Λ and the condition ζ_0 is allowed to be any probability measure. In that case the existence or the well-posedness will only be possible in the weak sense, i.e., not on a fixed (a priori) probability space.
The aim of this section is to show weak existence for problem (5.23), in the sense of Definition 2.7 under Assumption 3. The idea consists here in regularizing the functions Φ and g and truncating the initial condition ζ_0 to use existence result stated in Section 4, i.e. Theorem 4.2.

Theorem 5.1. *Under Assumption 3, the problem* (1.4) *admits existence in law, i.e. there is a solution* (Y,u) *of* (5.23) *on a suitable probability space equipped with some Brownian motion.*

Proof. We consider the following mollifications (resp. truncations) of the coefficients (resp. the initial condition).

$$\begin{array}{l} \Phi_n : (t,x,\xi)\in[0,T]\times\mathbb{R}^d\times\mathbb{R} \mapsto \int_{\mathbb{R}^d\times\mathbb{R}} \phi_n^d(x-r')\phi_n(\xi-r)\Phi(t,r',r)dr'dr \\ g_n : (t,x,\xi)\in[0,T]\times\mathbb{R}^d\times\mathbb{R} \mapsto \int_{\mathbb{R}^d\times\mathbb{R}} \phi_n^d(x-r')\phi_n(\xi-r)g(t,r',r)dr'dr \\ \forall\varphi\in\mathcal{C}_b(\mathbb{R}^d),\ \int_{\mathbb{R}^d}\zeta_0^n(dx)\varphi(x) = \int_{\mathbb{R}^d}\zeta_0(dx)\varphi(-n\vee x\wedge n) \ . \end{array} \tag{5.24}$$

We fix a filtered probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion W. First of all, we point out the fact that the function Λ satisfies the same assumptions as in Section 4. On one hand, by (5.24), since ϕ_n^d belongs to $\mathcal{S}(\mathbb{R}^d)$, Φ_n and g_n are uniformly bounded and Lipschitz with respect to (x,ξ) uniformly w.r.t. t for each $n\in\mathbb{N}$. Also ζ_0^n admits a second moment and (ξ_0^n) weakly converges to ξ_0. For each n, let Y_0^n be a (square integrable) r.v. distributed according to ζ_0^n. On the other hand, by Theorem 4.2, there is a pair (Y^n, u^n) fulfilling (1.4) with Φ, g, ζ_0 replaced by Φ_n, g_n, ζ_0^n. In particular we have

$$\left\{\begin{array}{l} Y_t^n = Y_0^n + \int_0^t \Phi_n(r, Y_r^n, u_n(r, Y_r^n))dW_r + \int_0^t g_n(r, Y_r^n, u_n(r, Y_r^n))dr \ , \quad \text{with} \quad Y_0^n \sim \zeta_0^n \ , \\ u_n(t,x) = \int_{\mathcal{C}^d} dm^n(\omega) \left[K(x - X_t(\omega)) \ \exp\left\{\int_0^t \Lambda\big(r, X_r(\omega), u_n(r, X_r(\omega))\big)dr\right\}\right] \ , \quad \text{for} \quad (t,x) \in [0,T] \times \mathbb{R}^d \ , \\ m^n = \mathcal{L}(Y^n). \end{array}\right. \tag{5.25}$$

Since $(\zeta_0^n)_{n\in\mathbb{N}}$ weakly converges to ζ_0, it is tight. Being Φ_n and g_n uniformly bounded, the hypotheses of Lemma 7.10 are fulfilled. So, setting $Z^n := u_n(\cdot, Y^n)$, that lemma implies that there is a sequence (Y^{n_k}, Z^{n_k}) converging in law. For simplicity we replace in the sequel the subsequence (n_k) by (n). Let (Y^n) be the sequence of processes solving (5.25). We recall that (m^n) denotes the sequence of their law. The final result will be established once we will have proved the following statements.

a) u^n converges to some (continuous) function $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$, uniformly on each compact of $[0,T]\times\mathbb{R}^d$, which verifies

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, \ u(t,x) = \int_{\mathcal{C}^d} K(x - X_t(\omega)) \ \exp\left\{\int_0^t \Lambda\big(r, X_r(\omega), u(r, X_r(\omega)))\big)dr\right\} dm(\omega),$$

where m is the limit of the laws of m^n.

b) The processes Y^n converge in law to Y, where Y is a solution, in law, of

$$\left\{\begin{array}{l} Y_t = Y_0 + \int_0^t \Phi(r, Y_r, u(r, Y_r))dW_r + \int_0^t g(r, Y_r, u(r, Y_r))dr \\ Y_0 \sim \zeta_0 \ . \end{array}\right. \tag{5.26}$$

Step a) is a consequence of Proposition 4.3 with for all $n \in \mathbb{N}, \ \Lambda_n = \Lambda$.
To prove the second step b), we will pass to the limit in the first equation of (5.25). To this end, let us designate by $C_0^2(\mathbb{R}^d)$, the space of $C^2(\mathbb{R}^d)$ functions with compact support. Without loss of generality, we suppose $d = 1$. We will prove that m is a solution to the martingale problem (in the sense of Stroock and Varadhan, see chapter 6 in [17]) associated with the first equation of (5.23). In fact we will show that

$$\left\{\begin{array}{l} \forall \varphi \in C_0^2(\mathbb{R}), t \in [0,T], \ M_t := \varphi(X_t) - \varphi(X_0) - \int_0^t (\mathcal{A}_r\varphi)(X_r)dr, \ \text{is a } \mathcal{F}_t^X\text{-martingale, where} \\ (\mathcal{F}_t^X, t \in [0,T]) \text{ is the canonical filtration generated by } X, \end{array}\right. \tag{5.27}$$

where we set $(\mathcal{A}_r\varphi)(x) = \frac{1}{2}\Phi^2(r, x, u(r,x)))\varphi''(x) + g(r, x, u(r,x))\varphi'(x), \ r \in [0,T], x \in \mathbb{R}$.
Let $0 \leq s < t \leq T$ fixed, $F : \mathcal{C}([0,s], \mathbb{R}) \to \mathbb{R}$ continuous and bounded. Indeed, we will show

$$\forall \varphi \in C_0^2(\mathbb{R}), \ \mathbb{E}^m\left[\left(\varphi(X_t) - \varphi(X_0) - \int_0^t (\mathcal{A}_r\varphi)(X_r)dr\right) F(X_r, r \leq s)\right] = 0 \tag{5.28}$$

We recall that, for $n \in \mathbb{N}$, by definition, m^n is the law of the strong solution Y^n of

$$Y_t^n = Y_0^n + \int_0^t \Phi_n(r, Y_r^n, u_n(r, Y_r^n))dW_r + \int_0^t g_n(r, Y_r^n, u_n(r, Y_r^n))dr \ ,$$

on a fixed underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with related expectation $\mathbb{E}$.
Then, by Itô's formula, we easily deduce that $\forall n \in \mathbb{N}$,

$$\mathbb{E}\left[\left(\varphi(Y_t^n) - \varphi(Y_s^n) - \int_s^t \left(\frac{1}{2}\Phi_n^2(r, Y_r^n, u_n(r, Y_r^n))\varphi''(Y_r^n) + g_n(r, Y_r^n, u_n(r, Y_r^n))\varphi'(Y_r^n)\right) dr\right) F(Y_r^n, r \leq s)\right] = 0 \, . \tag{5.29}$$

Transferring this to the canonical space $\mathcal{C}$ and to the probability m^n gives

$$\mathbb{E}^{m^n}\left[\left(\varphi(X_t)-\varphi(X_s)-\int_s^t\left(\frac{1}{2}\Phi_n^2(r,X_r,u_n(r,X_r))\varphi''(X_r)+g_n(r,X_r,u_n(r,X_r))\varphi'(X_r)\right)dr\right)F((X_u,0\leq u\leq s))\right]=0. \tag{5.30}$$

From now on, we are going to pass to the limit when $n\longrightarrow+\infty$ in (5.30) to obtain (5.27). Thanks to the weak convergence of the sequence m^n, for $\varphi\in C_0^2(\mathbb{R})$, we have immediately

$$\mathbb{E}^{m_n}[(\varphi(X_t)-\varphi(X_s))\,F(X_u,0\leq u\leq s)]-\mathbb{E}^m[(\varphi(X_t)-\varphi(X_s))\,F(X_u,0\leq u\leq s)]\xrightarrow[n\longrightarrow\infty]{}0. \tag{5.31}$$

It remains to show,

$$\begin{cases}\lim_{n\longrightarrow\infty}\mathbb{E}^{m_n}[H^n(X)F(X_u,0\leq u\leq s)]=\mathbb{E}^m[H(X)F(X_u,0\leq u\leq s)],\\ \text{with } H^n(\alpha):=\int_s^t(\frac{1}{2}\Phi_n^2(r,\alpha_r,u_n(r,\alpha_r))\varphi''(\alpha_r)+g_n(r,\alpha_r,u_n(r,\alpha_r))\varphi'(\alpha_r)dr,\\ H(\alpha):=\int_s^t(\frac{1}{2}\Phi^2(r,\alpha_r,u(r,\alpha_r))\varphi''(\alpha_r)+g(r,\alpha_r,u(r,\alpha_r))\varphi'(\alpha_r)dr\;.\end{cases} \tag{5.32}$$

In order to show that $\mathbb{E}^{m_n}[H^n(X)F(X)]-\mathbb{E}^m[H(X)F(X)]$ goes to zero, we will apply again Lemma 7.2. As we have mentioned above, F is continuous and bounded. Similarly as for Lemma 7.5, the proof of the continuity of H (resp. H_n) makes use of the continuity of Φ, g, φ'', φ' (resp. Φ_n, g_n,φ'', φ') and Lebesgue dominated convergence theorem.

Taking into account Remark 7.3, it is enough to prove the uniform convergence of $H^n:\mathcal{C}\longrightarrow\mathbb{R}$ to $H:\mathcal{C}\longrightarrow\mathbb{R}$ on each ball of $\mathcal{C}$. This relies on the uniform convergence of $\Phi_n(t,\cdot,\cdot)$ (resp. $g_n(t,\cdot,\cdot)$) to $\Phi(t,\cdot,\cdot)$ (resp. $g(t,\cdot,\cdot)$) on every compact subset $\mathbb{R}\times\mathbb{R}$, for fixed $t\in[0,T]$. Since the sequence (m^n) converges weakly, finally Lemma 7.2 allows to conclude (5.32).

□

6 Link with nonlinear Partial Differential Equation

From now on, in all the sequel, to simplify notations, we will often use the notation $f_t(\cdot)=f(t,\cdot)$ for functions $f:[0,T]\times E\longrightarrow\mathbb{R}$, E being some metric space.

In the following, we suppose again the validity of Assumption 3.
In this section, we want to link the nonlinear SDE (1.4) to a partial integro-differential equation (PIDE) that we have to determine. We start by considering problem (1.4) written under the form

$$\begin{cases}Y_t=Y_0+\int_0^t\Phi(s,Y_s,u_s^m(Y_s))dW_s+\int_0^t g(s,Y_s,u_s^m(Y_s))ds,\quad Y_0\sim\zeta_0\\ u_t^m(x)=\int_{\mathcal{C}^d}K(x-X_t(\omega))\,\exp\left\{\int_0^t\Lambda\big(r,X_r(\omega),u_r^m(X_r(\omega))\big)dr\right\}dm(\omega)\\ \mathcal{L}(Y)=m.\end{cases} \tag{6.1}$$

Suppose that K is formally the Dirac measure at zero and consider a solution (Y,v) of (6.1). We can easily show that v is a solution of (1.3) in the sense of distributions. Indeed let $\varphi\in\mathcal{S}(\mathbb{R}^d)$. Applying Itô formula to $\varphi(Y_t)$ we can easily show that the function v, which is the density of the measure ν defined in (1.3), is a solution in the sense of distributions of (1.1). For K being a mollifier of the Dirac measure, applying the same strategy, we cannot easily identify the deterministic problem (e.g. PDE or PIDE) solved by u^m.

For that reason we begin by establishing a correspondence between (6.1) and another McKean type stochastic differential equation, i.e.

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s, (K * \gamma^m)(s, Y_s))dW_s + \int_0^t g(s, Y_s, (K * \gamma^m)(s, Y_s))ds, \quad Y_0 \sim \zeta_0 \\ \gamma^m_t \text{ is the measure defined by, for all } \varphi \in \mathcal{C}_b(\mathbb{R}^d) \\ \gamma^m_t(\varphi) := \langle \gamma^m_t, \varphi \rangle := \int_{\mathcal{C}^d} \varphi(X_t(\omega)) V_t(X, (K * \gamma^m)(X)) dm(\omega) \\ \mathcal{L}(Y) = m \ , \end{cases} \tag{6.2}$$

where we recall the notations $(K * \gamma)(s, \cdot) := (K * \gamma_s)(\cdot)$ and $\gamma^m_t(\varphi) := \int_{\mathbb{R}^d} \varphi(x)\gamma^m_t(dx)$.

Theorem 6.1. *We suppose the validity of Assumption 3. The existence of the McKean type stochastic differential equation* (6.1) *is equivalent to the one of* (6.2). *More precisely, given a solution* (Y, γ^m) *of* (6.2), (Y, u^m), *with* $u^m = K * \gamma^m$, *is a solution of* (6.1) *and if* (Y, u^m) *is a solution of* (6.1), *there exists a measure valued function* γ^m *such that* (Y, γ^m) *is solution of* (6.2).
In addition, if the measurable set $\{\xi \in \mathbb{R}^d | \mathcal{F}(K)(\xi) = 0\}$ *is Lebesgue negligible,* (6.1) *and* (6.2) *are equivalent, i.e., the solution measure* γ^m *(of* (6.2)*) is uniquely determined by the solution function* u^m *((6.1)) and conversely. We recall that the map* $\mathcal{F}$ *denotes Fourier transform.*

Proof. Let (Y, u^m) be a solution of (6.1). Let us fix $t \in [0, T]$.
Since $K \in L^1(\mathbb{R}^d)$, the Fourier transform applied to the function $u^m(t, \cdot)$ gives

$$\mathcal{F}(u^m)(t, \xi) = \mathcal{F}(K)(\xi) \int_{\mathcal{C}^d} e^{-i\xi \cdot X_t(\omega)} \exp\left(\int_0^t \Lambda(r, X_r(\omega), u^m_r(X_r(\omega))) \right) dm(\omega) \ . \tag{6.3}$$

By Lebesgue dominated convergence theorem, one can show that the function

$$f^m : \xi \in \mathbb{R}^d \mapsto f^m(\xi) := \int_{\mathcal{C}^d} e^{-i\xi \cdot X_t(\omega)} \exp\left(\int_0^t \Lambda(r, X_r(\omega), u^m_r(X_r(\omega))) \right) dm(\omega) \ ,$$

is continuous. Since Λ is bounded, f^m is also bounded. Let $(a_k)_{k=1,\cdots,d}$ be a sequence of complex numbers and $(x_k)_{k=1\cdots,d} \in (\mathbb{R}^d)^d$. Remarking that for all $\xi \in \mathbb{R}^d$

$$\sum_{k=1}^d \sum_{p=1}^d a_k \bar{a}_p e^{-i\xi \cdot (x_k - x_p)} = \left(\sum_{k=1}^d a_k e^{-i\xi \cdot x_k} \right) \overline{\left(\sum_{p=1}^d a_p e^{-i\xi \cdot x_p} \right)} = \left| \sum_{k=1}^d a_k e^{-i\xi \cdot x_k} \right|^2 \ ,$$

which shows that f^m is non-negative definite. Then, by Bochner's theorem (see Theorem 24.9 Chapter I.24 in [16]), there exists a finite non-negative Borel measure μ_t on $\mathbb{R}^d$ such that for all $\xi \in \mathbb{R}^d$

$$f^m(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{-i\xi \cdot \theta} \mu^m_t(d\theta) \ . \tag{6.4}$$

We wish to show that $\gamma^m_t := \mu^m_t$ fulfills the third line equation of (6.2).
Since μ^m_t is a finite (non-negative) Borel measure, it is a Schwartz (tempered) distribution such that

$$\mathcal{F}^{-1}(f^m) = \mu^m_t \quad \text{ and } \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d), \ \left| \int_{\mathbb{R}^d} \psi(x) \mu^m_t(dx) \right| \leq \|\psi\|_\infty \mu^m_t(\mathbb{R}^d) < \infty \ .$$

On one hand, equalities (6.3) and (6.4) give

$$\mathcal{F}(u^m)(t, \cdot) = \mathcal{F}(K)\mathcal{F}(\mu^m_t) \Longrightarrow u^m(t, \cdot) = K * \mu^m_t \ . \tag{6.5}$$

On the other hand, for all $\psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
\langle \mu_t^m, \psi \rangle &= \langle \mathcal{F}^{-1}(f^m), \psi \rangle \\
&= \langle f^m, \mathcal{F}^{-1}(\psi) \rangle \\
&= \int_{\mathbb{R}^d} \mathcal{F}^{-1}(\psi)(\xi) \left(\int_{\mathcal{C}^d} e^{-i\xi \cdot X_t(\omega)} \exp(\int_0^t \Lambda(r, X_r(\omega), u_r^m(X_r(\omega)))) dm(\omega) \right) d\xi \\
&= \int_{\mathcal{C}^d} \left(\int_{\mathbb{R}^d} \mathcal{F}^{-1}(\psi)(\xi) e^{-i\xi \cdot X_t(\omega)} d\xi \right) \exp \left(\int_0^t \Lambda(r, X_r(\omega), u_r^m(X_r(\omega))) \right) dm(\omega) \\
&= \int_{\mathcal{C}^d} \left(\int_{\mathbb{R}^d} \mathcal{F}^{-1}(\psi)(\xi) e^{-i\xi \cdot X_t(\omega)} d\xi \right) \exp \left(\int_0^t \Lambda(r, X_r(\omega), (K * \mu_r^m)(X_r(\omega))) \right) dm(\omega) \\
&= \int_{\mathcal{C}^d} \psi(X_t(\omega)) \exp \left(\int_0^t \Lambda(r, X_r(\omega), (K * \mu_r^m)(X_r(\omega))) \right) dm(\omega) ,
\end{aligned}$$

where the fourth equality is justified by Fubini theorem and the fifth equality follows by (6.5). This allows to conclude the necessary part of the first lemma statement.

Regarding the converse, let (Y, γ^m) be a solution of (6.2). We set $u_t^m(x) := (K * \gamma_t^m)(x)$, so that the first equation in (6.1) is satisfied for (Y, u^m). Since μ_t^m is finite, the second equation follows directly setting $\varphi = K(x - \cdot)$ in (6.2).
To establish the second statement of the theorem, it is enough to observe that from the r.h.s. of (6.5) we have

$$\text{Leb}(\{\xi \in \mathbb{R}^d | \mathcal{F}(K)(\xi) = 0\}) = 0 \Longrightarrow \mathcal{F}(\mu_t^m) = \frac{\mathcal{F}(u^m)(t, \cdot)}{\mathcal{F}(K)} \text{ a.e. } , t \in [0, T],$$

where Leb denotes the Lebesgue measure on $\mathbb{R}^d$. This shows effectively that γ^m (resp. u^m) is uniquely determined by u^m (resp. γ^m) and ends the proof. □

Now, by applying Itô's formula , we can show that the associated measure γ^m (second equation in (6.2)) satisfies a PIDE.

Theorem 6.2. *The measure γ_t^m, defined in the second equation of* (6.2)*, satisfies the PIDE*

$$\begin{cases} \partial_t \gamma_t^m &= \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2 \left((\Phi\Phi^t)_{i,j}(t, x, (K * \gamma_t^m)) \gamma_t^m \right) - div \left(g(t, x, K * \gamma_t^m) \gamma_t^m \right) + \gamma_t^m \Lambda(t, x, (K * \gamma_t^m)) \\ \gamma_0^m(dx) &= \zeta_0(dx) , \end{cases} \tag{6.6}$$

in the sense of distributions, i.e. for every $t \in [0, T], \varphi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi(x) \gamma_t^m(dx) &= \int_{\mathbb{R}^d} \varphi(x) \zeta_0(dx) \\
&+ \int_0^t \int_{\mathbb{R}^d} \varphi(x) \Lambda(s, x, (K * \gamma^m)(s, x)) \gamma_s^m(dx) ds \\
&+ \int_0^t \int_{\mathbb{R}^d} \nabla \varphi(x) \cdot g(s, x, (K * \gamma^m)(s, x)) \gamma_s^m(dx) ds dx \\
&+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \int_{\mathbb{R}^d} \partial_{ij}^2 \varphi(x) (\Phi\Phi^t)_{i,j}(s, x, (K * \gamma^m)(s, x)) \gamma_s^m(dx) ds dx .
\end{aligned} \tag{6.7}$$

Proof. It is enough to use the definition of γ_t^m and, as mentioned above, apply Itô's formula to the process $\varphi(Y_t) V_t(Y, (K * \gamma^m)(Y))$, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and Y (defined in the first equation of (6.2)). Indeed, for $\varphi \in$

$\mathcal{C}_0^\infty(\mathbb{R}^d)$, Itô's formula gives

$$\begin{aligned}
\mathbb{E}[\varphi(Y_t)V_t(Y,(K*\gamma^m)(Y))] &= \mathbb{E}[\varphi(Y_0)] \\
&+ \int_0^t \mathbb{E}\left[\varphi(Y_s)\Lambda(s,Y_s,(K*\gamma^m)(s,Y_s))V_s(Y,(K*\gamma^m)(Y))\right] ds \\
&+ \int_0^t \sum_{i=1}^d \mathbb{E}\left[\partial_i\varphi(Y_s)g_i(s,Y_s,(K*\gamma^m)(s,Y_s))V_s(Y,(K*\gamma^m)(Y))\right] ds \\
&+ \frac{1}{2}\int_0^t \sum_{i,j=1}^d \mathbb{E}[\partial^2_{ij}\varphi(Y_s)(\Phi\Phi^t)_{i,j}(s,Y_s,(K*\gamma^m)(s,Y_s))V_s(Y,(K*\gamma^m)(Y))]ds\,.
\end{aligned}$$

By the definition of the measure γ_t^m, for each $t \in [0,T]$, in (6.2), we have

$$\begin{aligned}
\int_{\mathbb{R}^d}\varphi(x)\gamma_t^m(dx) &= \int_{\mathbb{R}^d}\varphi(x)\zeta_0(dx) \\
&+ \int_0^t\int_{\mathbb{R}^d}\varphi(x)\Lambda(s,x,(K*\gamma^m)(s,x))\gamma_s^m(dx)ds \\
&+ \int_0^t\int_{\mathbb{R}^d}\nabla\varphi(x)\cdot g(s,x,(K*\gamma^m)(s,x))\gamma_s^m(dx)ds \\
&+ \frac{1}{2}\sum_{i,j=1}^d\int_0^t\int_{\mathbb{R}^d}\partial^2_{ij}\varphi(x)(\Phi\Phi^t)_{i,j}(s,x,(K*\gamma^m)(s,x))\gamma_s^m(dx)ds\,.
\end{aligned}$$

This concludes the proof of Theorem 6.2. □

7 Appendix

In this appendix, we present the proof of some technical results used in previous sections.

7.1 Proofs of the technicalities related to Section 3.1

In this section, we prove the fundamental properties of the map $(m,t,x) \to u^m(t,x)$ announced in Proposition 3.3.

Proof of Proposition 3.3. We will prove successively the inequalities (3.6), (3.7), (3.8) and (3.9).
Let us consider $(t,y,y') \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$.

- **Proof of** (3.6) . Let $(m,m') \in \mathcal{P}_2(\mathcal{C}^d)\times\mathcal{P}_2(\mathcal{C}^d)$.
  We have

$$|u^m(t,y)-u^{m'}(t,y')|^2 \leq 2|u^m(t,y)-u^m(t,y')|^2+2|u^m(t,y')-u^{m'}(t,y')|^2\,. \tag{7.1}$$

  The first term on the r.h.s. of the above equality is bounded using the Lipschitz property of u^m that derives straightforwardly from the Lipschitz property of the mollifier K and the boundedness property of V_t (2.6):

$$\begin{aligned}
|u^m(t,y')-u^m(t,y)| &= \left|\int_{\mathcal{C}^d}\left[K(y-X_t(\omega))-K(y'-X_t(\omega))\right]V_t(X(\omega),u^m(X(\omega)))dm(\omega)\right| \\
&\leq L_K e^{tM_\Lambda}|y-y'|\,. \qquad (7.2)
\end{aligned}$$

Now let us consider the second term on the r.h.s. of (7.1). By Jensen's inequality we get

$$
\begin{aligned}
|u^m(t,y') - u^{m'}(t,y')|^2 &= \Big| \int_{\mathcal{C}^d} K(y' - X_t(\omega)) V_t\big(X(\omega), u^m(X(\omega))\big) dm(\omega) \\
&\quad - \int_{\mathcal{C}^d} K(y' - X_t(\omega')) V_t\big(X(\omega'), u^{m'}(X(\omega'))\big) dm'(\omega') \Big|^2 \\
&\leq \int_{\mathcal{C}^d \times \mathcal{C}^d} \Big| K(y' - X_t(\omega)) V_t\big(X(\omega), u^m(X(\omega))\big) \\
&\quad - K(y' - X_t(\omega')) V_t\big(X(\omega'), u^{m'}(X(\omega'))\big) \Big|^2 d\mu(\omega,\omega') \,, \qquad (7.3)
\end{aligned}
$$

for any $\mu \in \Pi(m, m')$. Let us consider four continuous functions $x,\, x' \in \mathcal{C}([0,T], \mathbb{R}^d)$ and $z, z' \in \mathcal{C}([0,T], \mathbb{R})$. We have

$$
\begin{aligned}
|K(y' - x_t)V_t(x,z) - K(y' - x'_t)V_t(x',z')|^2 &\leq 2\,|K(y' - x_t) - K(y' - x'_t)|^2\, |V_t(x,z)|^2 \\
&\quad + 2\,|V_t(x,z) - V_t(x',z')|^2\, |K(y' - x'_t)|^2 \,.
\end{aligned}
$$

Then, using the Lipschitz property of K and the upper bound (2.8) gives

$$
\begin{aligned}
|K(y' - x_t)V_t(x,z) - K(y' - x'_t)V_t(x',z')|^2 &\leq 2L_K^2 e^{2tM_\Lambda} |x_t - x'_t|^2 \\
&\quad + 4M_K^2 L_\Lambda^2 e^{2tM_\Lambda} t \int_0^t \left[|x_s - x'_s|^2 + |z_s - z'_s|^2\right] ds \qquad (7.4) \\
&\leq C'_{K,\Lambda}(t) \left[(1+t) \sup_{s\leq t} |x_s - x'_s|^2 + \int_0^t |z_s - z'_s|^2\, ds\right] ,
\end{aligned}
$$

where $C'_{K,\Lambda}(t) = 2e^{2tM_\Lambda}(L_K^2 + 2M_K^2 L_\Lambda^2 t)$. Injecting the latter inequality in (7.3) yields

$$
\begin{aligned}
|u^m(t,y') - u^{m'}(t,y')|^2 &\leq C'_{K,\Lambda}(t) \int_{\mathcal{C}^d \times \mathcal{C}^d} \Big[(1+t) \sup_{s\leq t} |X_s(\omega) - X_s(\omega')|^2 \\
&\quad + \int_0^t |u^m(s, X_s(\omega)) - u^{m'}(s, X_s(\omega'))|^2\, ds\Big]\, d\mu(\omega,\omega') \,.
\end{aligned}
$$

Injecting the above inequality in (7.1) and using (7.2) yields

$$
\begin{aligned}
|u^m(t,y) - u^{m'}(t,y')|^2 &\leq 2C'_{K,\Lambda}(t) \Big[|y - y'|^2 + (1+t) \int_{\mathcal{C}^d \times \mathcal{C}^d} \sup_{s\leq t} |X_s(\omega) - X_s(\omega')|^2 d\mu(\omega,\omega') \\
&\quad + \int_{\mathcal{C}^d \times \mathcal{C}^d} \int_0^t |u^m(s, X_s(\omega)) - u^{m'}(s, X_s(\omega'))|^2\, ds\, d\mu(\omega,\omega')\Big] ,
\end{aligned}
$$

Replacing y (resp. y') with $X_t(\omega)$ (resp. $X_t(\omega')$) in (7.5), we get for all $\omega \in \mathcal{C}^d$ (resp. $\omega' \in \mathcal{C}^d$),

$$
\begin{aligned}
|u^m(t, X_t(\omega)) - u^{m'}(t, X_t(\omega'))|^2 &\leq 2C'_{K,\Lambda}(t) \Big[|X_t(\omega) - X_t(\omega')|^2 \\
&\quad + (1+t) \int_{\mathcal{C}^d \times \mathcal{C}^d} \sup_{s\leq t} |X_s(\omega) - X_s(\omega')|^2 d\mu(\omega,\omega') \\
&\quad + \int_{\mathcal{C}^d \times \mathcal{C}^d} \int_0^t |u^m(s, X_s(\omega)) - u^{m'}(s, X_s(\omega'))|^2\, ds\, d\mu(\omega,\omega')\Big].
\end{aligned}
$$

Let us introduce the notation

$$
\gamma(s) := \int_{\mathcal{C}^d \times \mathcal{C}^d} |u^m(s, X_s(\omega)) - u^{m'}(s, X_s(\omega'))|^2\, d\mu(\omega,\omega') \,, \quad \text{for any } s \in [0,T].
$$

Integrating each side of inequality (7.5) w.r.t. the variables (ω,ω') according to μ, implies

$$\gamma(t) \leq 2C'_{K,\Lambda}(t)\int_0^t \gamma(s)ds + 2(t+2)C'_{K,\Lambda}(t)\int_{\mathcal{C}^d\times\mathcal{C}^d}\sup_{s\leq t}|X_s(\omega)-X_s(\omega')|^2 d\mu(\omega,\omega')\ ,$$

for all $t\in[0,T]$. In particular, observing that $C'_{K,\Lambda}(a)$ is increasing in a, we have for fixed $t\in]0,T]$ and all $a\in[0,t]$

$$\gamma(a) \leq 2C'_{K,\Lambda}(t)\int_0^a \gamma(s)ds + 2(t+2)C'_{K,\Lambda}(t)\int_{\mathcal{C}^d\times\mathcal{C}^d}\sup_{s\leq t}|X_s(\omega)-X_s(\omega')|^2 d\mu(\omega,\omega')\ .$$

Using Gronwall's lemma yields

$$\begin{aligned}\gamma(t) \quad &:= \quad \int_{\mathcal{C}^d\times\mathcal{C}^d}|u^m(t,X_t(\omega))-u^{m'}(t,X_t(\omega'))|^2\, d\mu(\omega,\omega') \\ &\leq \quad 2(t+2)C'_{K,\Lambda}(t)e^{2tC'_{K,\Lambda}(t)}\int_{\mathcal{C}^d\times\mathcal{C}^d}\sup_{s\leq t}|X_s(\omega)-X_s(\omega')|^2 d\mu(\omega,\omega')\ .\end{aligned}$$

Injecting the above inequality in (7.5) implies

$$|u^m(t,y)-u^{m'}(t,y')|^2 \leq 2C'_{K,\Lambda}(t)(t+2)(1+e^{2tC'_{K,\Lambda}(t)})\left[|y-y'|^2+\int_{\mathcal{C}^d\times\mathcal{C}^d}\sup_{s\leq t}|X_s(\omega)-X_s(\omega')|^2 d\mu(\omega,\omega')\right]\ . \tag{7.5}$$

The above inequality holds for any $\mu\in\Pi(m,m')$, hence taking the infimum over $\mu\in\Pi(m,m')$ concludes the proof of (3.6).

- **Proof of** (3.7). Let $(m,m')\in\mathcal{P}(\mathcal{C}^d)\times\mathcal{P}(\mathcal{C}^d)$. The proof of (3.7) follows at the beginning the same lines as the one of (3.6), but the inequality (7.4) is replaced by

$$\begin{aligned}|K(y'-x_t)V_t(x,z)-K(y'-x'_t)V_t(x',z')|^2 \quad &\leq \quad 2\,|K(y'-x_t)-K(y'-x'_t)|^2\,|V_t(x,z)|^2 \\ &\quad\ +2\,|V_t(x,z)-V_t(x',z')|^2\,|K(y'-x'_t)|^2 \\ &\leq \quad 2e^{2tM_\Lambda}\max(L_K,2M_K)^2(|x_t-x'_t|^2\wedge 1) \\ &\quad\ +4M_K^2e^{2tM_\Lambda}\max(L_\Lambda,2M_\Lambda)^2 t\int_0^t\Big(|x'_s-x_s|^2\wedge 1 \\ &\quad\ +|z_s-z'_s|^2\Big)ds \\ &\leq \quad \mathfrak{C}'_{K,\Lambda}(t)\left[(1+t)(\sup_{s\leq t}|x_s-x'_s|^2\wedge 1)+\int_0^t|z_s-z'_s|^2\,ds\right],\end{aligned}$$

where $\mathfrak{C}'_{K,\Lambda}(t) := 2e^{2tM_\Lambda}(\max(L_K,2M_K)^2+2M_K^2\max(L_\Lambda,2M_\Lambda)^2t)$. Following the same lines as for the proof of item 1. leads to

$$\begin{aligned}|u^m(t,y)-u^{m'}(t,y')|^2 \quad &\leq \quad 2\mathfrak{C}'_{K,\Lambda}(t)(t+2)(1+e^{2t\mathfrak{C}'_{K,\Lambda}(t)})\Big[\,|y-y'|^2 \\ &\quad\ +\int_{\mathcal{C}^d\times\mathcal{C}^d}\sup_{s\leq t}|X_s(\omega)-X_s(\omega')|^2\wedge 1\ d\mu(\omega,\omega')\Big]\ ,\end{aligned}$$

which constitutes the analogue of (7.5) and we conclude in the same way as for the previous item.

- **Proof of the continuity of** $(m,t,x)\mapsto u^m(t,x)$**.**
$\mathcal{P}(\mathcal{C}^d)\times[0,T]\times\mathbb{R}^d$ being a separable metric space, we characterize the continuity through converging

sequences. We also recall that $\widetilde{\mathcal{W}}_T$ is a distance compatible with the weak convergence on $\mathcal{P}(\mathcal{C}^d)$, see Remark 3.4 a).

By (3.7), the application is continuous with respect to (m,x) uniformly with respect to time. Consequently it remains to show that the map $t \mapsto u^m(t,x)$ is continuous for fixed $(m,x) \in \mathcal{P}(\mathcal{C}^d) \times \mathbb{R}^d$.

Let us fix $(m,t_0,x) \in \mathcal{P}(\mathcal{C}^d) \times [0,T] \times \mathbb{R}^d$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0,T]$ converging to t_0.

We define F_n as the real-valued sequence of measurable functions on $\mathcal{C}^d$ such that for all $\omega \in \mathcal{C}^d$,

$$F_n(\omega) := K(x - X_{t_n}(\omega)) \exp\left(\int_0^{t_n} \Lambda(r, X_r(\omega), u^m(r, X_r(\omega))dr\right) . \tag{7.6}$$

Each $\omega \in \mathcal{C}^d$ being continuous, F_n converges pointwise to $F : \mathcal{C}^d \to \mathbb{R}$ defined by

$$F(\omega) := K(x - X_{t_0}(\omega)) \exp\left(\int_0^{t_0} \Lambda(r, X_r(\omega), u^m(r, X_r(\omega))dr\right) . \tag{7.7}$$

Since K and Λ are uniformly bounded, $M_K e^{TM_\Lambda}$ is a uniform upper bound of the functions F_n. By Lebesgue dominated convergence theorem, we conclude that

$$|u^m(t_n,x) - u^m(t_0,x)| = \left|\int_{\mathcal{C}^d} F_n(\omega)dm(\omega) - \int_{\mathcal{C}^d} F(\omega)dm(\omega)\right| \xrightarrow[n\to+\infty]{} 0 .$$

This ends the proof.

- **Proof of** (3.8). Let $(m,m') \in \mathcal{P}_2(\mathcal{C}^d) \times \mathcal{P}_2(\mathcal{C}^d)$.
Since $K \in L^2(\mathbb{R}^d)$, by Jensen's inequality, it follows easily that the functions $x \mapsto u^m(r,x)$ and $x \mapsto u^{m'}(r,x)$ belong to $L^2(\mathbb{R}^d)$, for every $r \in [0,T]$. Then, for any $\mu \in \Pi(m,m')$,

$$\begin{aligned}
\|u^m(t,\cdot) - u^{m'}(t,\cdot)\|_2^2 &= \int_{\mathbb{R}^d} |u^m(t,y) - u^{m'}(t,y)|^2 dy \\
&= \int_{\mathbb{R}^d} \left|\int_{\mathcal{C}^d\times\mathcal{C}^d} \Big[K(y - X_t(\omega))\, V_t(X(\omega), u^m(X(\omega))) - \right. \\
&\qquad \left. K(y - X_t(\omega'))\, V_t(X(\omega'), u^{m'}(X(\omega')))\Big]\, d\mu(\omega,\omega') \right|^2 dy \\
&\le \int_{\mathbb{R}^d} \int_{\mathcal{C}^d\times\mathcal{C}^d} \Big|K(y - X_t(\omega))\, V_t(X(\omega), u^m(X(\omega))) - \\
&\qquad K(y - X_t(\omega'))\, V_t(X(\omega'), u^{m'}(X(\omega')))\Big|^2 d\mu(\omega,\omega')\, dy \\
&= \int_{\mathcal{C}^d\times\mathcal{C}^d} \int_{\mathbb{R}^d} \Big|K(y - X_t(\omega))\, V_t(X(\omega), u^m(X(\omega))) - \\
&\qquad K(y - X_t(\omega'))\, V_t(X(\omega'), u^{m'}(X(\omega')))\Big|^2 dy\, d\mu(\omega,\omega') ,
\end{aligned} \tag{7.8}$$

where the third inequality follows by Jensen's and the latter equality is justified by Fubini theorem.
We integrate now both sides of (7.4), with respect to the state variable y over $\mathbb{R}^d$, for all $(x,x') \in \mathcal{C}^d \times \mathcal{C}^d, (z,z') \in \mathcal{C} \times \mathcal{C}$,

$$\begin{aligned}
\int_{\mathbb{R}^d} |K(y - x_t)V_t(x,z) - K(y - x'_t)V_t(x',z')|^2\, dy \le 2 \int_{\mathbb{R}^d} |K(y - x_t) - K(y - x'_t)|^2 |V_t(x,z)|^2\, dy \\
+2 \int_{\mathbb{R}^d} |V_t(x,z) - V_t(x',z')|^2 |K(y - x'_t)|^2\, dy.
\end{aligned} \tag{7.9}$$

We remark now that, by classical properties of Fourier transform, since $K \in L^2(\mathbb{R}^d)$, we have

$$\forall\, (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,\ \mathcal{F}(K_x)(\xi) = e^{-i\xi\cdot x}\mathcal{F}(K)(\xi) ,$$

where in this case, the Fourier transform operator $\mathcal{F}$ acts from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ and $K_x : \bar{y} \in \mathbb{R}^d \mapsto K(\bar{y}-x)$. Since $K \in L^2(\mathbb{R}^d)$, Plancherel's theorem gives, for all $(\bar{y}, x, x') \in \mathbb{R}^d \times \mathcal{C}^d \times \mathcal{C}^d$,

$$
\begin{aligned}
\int_{\mathbb{R}^d} |K(\bar{y}-x_t) - K(\bar{y}-x'_t)|^2 d\bar{y} &= \int_{\mathbb{R}^d} |K_{x_t}(\bar{y}) - K_{x'_t}(\bar{y})|^2 d\bar{y} \\
&= \int_{\mathbb{R}^d} |e^{-i\xi\cdot x_t}\mathcal{F}(K)(\xi) - e^{-i\xi\cdot x'_t}\mathcal{F}(K)(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)|^2 \; |e^{-i\xi\cdot x_t} - e^{-i\xi\cdot x'_t}|^2 d\xi \\
&\leq \int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)|^2 \; |\xi\cdot(x_t - x'_t)|^2 d\xi \\
&\leq |x_t - x'_t|^2 \int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)|^2 \; |\xi|^2 d\xi \\
&= |x_t - x'_t|^2 \int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)\xi|^2 d\xi \\
&= |x_t - x'_t|^2 \int_{\mathbb{R}^d} |\mathcal{F}(\nabla K)(\xi)|^2 d\xi \\
&= |x_t - x'_t|^2 \|\nabla K\|_2^2 \, .
\end{aligned}
\tag{7.10}
$$

Injecting this bound into (7.9), taking into account (2.8), yields

$$
\begin{aligned}
\int_{\mathbb{R}^d} |K(y-x_t)V_t(x,z) - K(y-x'_t)V_t(x',z')|^2 \, dy &\leq 2\|\nabla K\|_2^2 \, |x_t - x'_t|^2 \exp(2tM_\Lambda) \\
&\quad + 2M_K |V_t(x,z) - V_t(x',z')|^2 \\
&\leq 2e^{2tM_\Lambda}\|\nabla K\|_2^2 |x_t - x'_t|^2 \\
&\quad +4M_K L_\Lambda^2 e^{2tM_\Lambda} t \int_0^t \left[|x_s - x'_s|^2 + |z_s - z'_s|^2\right] ds \\
&\leq 2e^{2tM_\Lambda}(2M_K L_\Lambda^2 t^2 + \|\nabla K\|_2^2) \sup_{0\leq r\leq t} |x_r - x'_r|^2 \\
&\quad +4M_K L_\Lambda^2 e^{2tM_\Lambda} t \int_0^t |z_s - z'_s|^2 \, ds \\
&\leq \tilde{C}_{K,\Lambda}(t) \left[\sup_{0\leq r\leq t} |x_r - x'_r|^2 + \int_0^t |z_s - z'_s|^2 \, ds\right] ,
\end{aligned}
\tag{7.11}
$$

for all $(x, x') \in \mathcal{C}^d \times \mathcal{C}^d$ and $(z, z') \in \mathcal{C} \times \mathcal{C}$, with $\tilde{C}_{K,\Lambda}(t) := 2e^{2tM_\Lambda}(2M_K L_\Lambda^2 t(t+1) + \|\nabla K\|_2^2)$. Inserting (7.11) into (7.8), after substituting $X(\omega)$ with x, $X(\omega')$ with x', z with $u^m(X(\omega))$ and z' with $u^{m'}(X(\omega'))$, for any $\mu \in \Pi(m, m')$, we obtain the inequality

$$
\begin{aligned}
\|u^m(t,\cdot) - u^{m'}(t,\cdot)\|_2^2 &\leq \tilde{C}_{K,\Lambda}(t) \Big\{ \int_{\mathcal{C}^d\times\mathcal{C}^d} \sup_{0\leq r\leq t} |X_r(\omega) - X_r(\omega')|^2 \, d\mu(\omega,\omega') \\
&\quad + \int_{\mathcal{C}^d\times\mathcal{C}^d} \int_0^t |u^m(s, X_s(\omega)) - u^{m'}(s, X_s(\omega'))|^2 \, ds \, d\mu(\omega,\omega') \Big\} \, .
\end{aligned}
\tag{7.12}
$$

Since inequality (3.6) is verified for all $y \in \mathbb{R}^d$, $s \in [0,T]$, we obtain for all $\omega, \omega' \in \mathcal{C}^d$

$$
\begin{aligned}
|u^m\big(s, X_s(\omega)\big) - u^{m'}\big(s, X_s(\omega')\big)|^2 &\leq C_{K,\Lambda}(s) \left[|X_s(\omega) - X_s(\omega')|^2 \; + |\mathcal{W}_s(m,m')|^2\right] \\
&\leq C_{K,\Lambda}(s) \left[\sup_{0\leq r\leq s} |X_r(\omega) - X_r(\omega')|^2 + |\mathcal{W}_s(m,m')|^2\right] .
\end{aligned}
$$

Integrating each side of the above inequality with respect to the time variable s and the measure $\mu \in \Pi(m, m')$ and observing that $C_{K,\Lambda}(s)$ is increasing in s, yields

$$
\begin{aligned}
I &:= \int_{\mathcal{C}^d \times \mathcal{C}^d} \int_0^t |u^m(s, X_s(\omega)) - u^{m'}(s, X_s(\omega'))|^2 \, ds \, d\mu(\omega, \omega') \\
&\leq C_{K,\Lambda}(t) t \left[\int_{\mathcal{C}^d \times \mathcal{C}^d} \sup_{0 \leq r \leq t} |X_r(\omega) - X_r(\omega')|^2 \, d\mu(\omega, \omega') + |\mathcal{W}_t(m, m')|^2 \right] . \qquad (7.13)
\end{aligned}
$$

By injecting inequality (7.13) in the right-hand side of inequality (7.12), we obtain

$$
\begin{aligned}
\|u^m(t, \cdot) - u^{m'}(t, \cdot)\|_2^2 &\leq \tilde{C}_{K,\Lambda}(t)(1 + tC_{K,\Lambda}(t)) \int_{\mathcal{C}^d \times \mathcal{C}^d} \sup_{0 \leq r \leq t} |X_r(\omega) - X_r(\omega')|^2 d\mu(\omega, \omega') \\
&\quad + t\tilde{C}_{K,\Lambda}(t) C_{K,\Lambda}(t) |\mathcal{W}_t(m, m')|^2 . \qquad (7.14)
\end{aligned}
$$

By taking the infimum over $\mu \in \Pi(m, m')$ on the right-hand side, we obtain

$$
\|u^m(t, \cdot) - u^{m'}(t, \cdot)\|_2^2 \leq \tilde{C}_{K,\Lambda}(t)(1 + 2tC_{K,\Lambda}(t)) |\mathcal{W}_t(m, m')|^2 . \qquad (7.15)
$$

- **Proof of** (3.9).

By the hypothesis 4. in Assumption 1, $K \in L^1(\mathbb{R}^d)$. Given a function $g : [0, T] \times \mathbb{R}^d \to \mathbb{C}$, $(s, x) \mapsto g(s, x)$, its Fourier transform in the space variable x will be denoted by $(s, \xi) \mapsto \mathcal{F}(g)(s, \xi)$ instead of $\mathcal{F}g(s, \cdot)(\xi)$. Then for $(\bar{\omega}, s, \xi) \in \Omega \times [0, T] \times \mathbb{R}^d$, the Fourier transform of the functions $u^{\eta_{\bar{\omega}}}$ and u^m are given by

$$
\begin{aligned}
\mathcal{F}(u^{\eta_{\bar{\omega}}})(s, \xi) &= \mathcal{F}(K)(\xi) \int_{\mathcal{C}^d} e^{-i\xi \cdot X_s(\omega)} \exp\left(\int_0^s \Lambda\big(r, X_r(\omega), u^{\eta_{\bar{\omega}}}(r, X_r(\omega))\big) dr \right) d\eta_{\bar{\omega}}(\omega) \quad (7.16) \\
\mathcal{F}(u^m)(s, \xi) &= \mathcal{F}(K)(\xi) \int_{\mathcal{C}^d} e^{-i\xi \cdot X_s(\omega)} \exp\left(\int_0^s \Lambda\big(r, X_r(\omega), u^m(r, X_r(\omega))\big) dr \right) dm(\omega) . \quad (7.17)
\end{aligned}
$$

To simplify notations in the sequel, we will often use the convention

$$
V_r^\nu(y) := V_r(y, u^\nu(y)) = \exp\left(\int_0^r \Lambda\big(\theta, y_\theta, u^\nu(\theta, y_\theta)\big) d\theta \right) ,
$$

where u^ν is defined in (3.1), with $m = \nu$.

In this way, relations (7.16) and (7.17) can be re-written as

$$
\begin{aligned}
\mathcal{F}(u^{\eta_{\bar{\omega}}})(s, \xi) &= \mathcal{F}(K)(\xi) \int_{\mathcal{C}^d} e^{-i\xi \cdot X_s(\omega)} V_s^{\eta_{\bar{\omega}}}(X(\omega)) d\eta_{\bar{\omega}}(\omega) \\
\mathcal{F}(u^m)(s, \xi) &= \mathcal{F}(K)(\xi) \int_{\mathcal{C}^d} e^{-i\xi \cdot X_s(\omega)} V_s^m(X(\omega)) dm(\omega) ,
\end{aligned} \qquad (7.18)
$$

for $(\bar{\omega}, s, \xi) \in \Omega \times [0, T] \times \mathbb{R}^d$.

For a function $f \in L^1(\mathbb{R}^d)$ such that $\mathcal{F}(f) \in L^1(\mathbb{R}^d)$, the inversion formula of the Fourier transform is valid and implies

$$
f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \mathcal{F}(f)(\xi) e^{i\xi \cdot x} d\xi, \ x \in \mathbb{R}^d . \qquad (7.19)
$$

f is obviously bounded and continuous taking into account Lebesgue dominated convergence theorem. Moreover

$$
\|f\|_\infty \leq \frac{1}{\sqrt{2\pi}} \|\mathcal{F}(f)\|_1, \qquad (7.20)
$$

where we recall that $\|\cdot\|_1$ denotes the $L^1(\mathbb{R}^d)$-norm. As $\mathcal{F}(K)$ belongs to $L^1(\mathbb{R}^d)$, from (7.20) applied to the function $f = u^{\eta_{\bar\omega}}(s,\cdot) - u^m(s,\cdot)$ with fixed $\bar\omega \in \Omega$, $s \in [0,T]$, we get

$$\begin{aligned}\mathbb{E}[\|u^\eta(s,\cdot) - u^m(s,\cdot)\|_\infty^2] &\leq \frac{1}{\sqrt{2\pi}}\mathbb{E}[\|\mathcal{F}(u^\eta)(s,\cdot) - \mathcal{F}(u^m)(s,\cdot)\|_1^2] \\ &\leq \frac{1}{\sqrt{2\pi}}\mathbb{E}\left[\left(\int_{\mathbb{R}^d} |\mathcal{F}(u^{\eta_{\bar\omega}})(s,\xi) - \mathcal{F}(u^m)(s,\xi)|d\xi\right)^2\right], \qquad (7.21)\end{aligned}$$

where we recall that $\mathbb{E}$ is taken w.r.t. to $d\mathbb{P}(\bar\omega)$.

The terms intervening in the expression above are measurable. This can be justified by Fubini-Tonelli theorem and the fact that $(\bar\omega, s, x) \mapsto u^{\eta_{\bar\omega}}(s,x)$ is measurable from $(\Omega \times [0,T] \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We prove the latter point. By item 3. of this Lemma, we recall that the function $(m,t,x) \mapsto u^m(t,x)$ is continuous on $\mathcal{P}(\mathcal{C}^d) \times [0,T] \times \mathbb{R}^d$ and so measurable from $(\mathcal{P}(\mathcal{C}^d) \times [0,T] \times \mathbb{R}^d, \mathcal{B}(\mathcal{P}(\mathcal{C}^d)) \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The application $(\bar\omega, t, x) \mapsto (\eta_{\bar\omega}, t, x)$ being measurable from $(\Omega \times [0,T] \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}([0,T] \otimes \mathcal{B}(\mathbb{R}^d)))$ to $(\mathcal{P}(\mathcal{C}^d) \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d))$, by composition the map $(\bar\omega, s, x) \mapsto u^{\eta_{\bar\omega}}(s,x)$ is measurable. By Fubini-Tonelli theorem $(\bar\omega, s, \xi) \mapsto \mathcal{F}(u^{\eta_{\bar\omega}})(s,\xi))$ is measurable from $(\Omega \times [0,T] \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d)$ to $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ and $(s,\xi) \mapsto u^m(s,\xi)$ is measurable from $([0,T] \times \mathbb{R}^d, \mathcal{B}([0,T] \otimes \mathbb{R}^d)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

We are now ready to bound the right-hand side of (7.21). For all $(\bar\omega, s) \in \Omega \times [0,T]$, by (7.18)

$$\begin{aligned}|\mathcal{F}(u^m)(s,\xi) - \mathcal{F}(u^{\eta_{\bar\omega}})(s,\xi)| &\leq |\mathcal{F}(K)(\xi)|\left|\int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} V_s^m(X(\omega))dm(\omega) - \int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} V_s^m(X(\omega))d\eta_{\bar\omega}(\omega)\right| \\ &\quad + |\mathcal{F}(K)(\xi)|\left|\int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} V_s^m(X(\omega))d\eta_{\bar\omega}(\omega) - \int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} V_s^{\eta_{\bar\omega}}(X(\omega))d\eta_{\bar\omega}(\omega)\right|, \qquad (7.22)\end{aligned}$$

which implies

$$\begin{aligned}\left(\int_{\mathbb{R}^d} |\mathcal{F}(u^{\eta_{\bar\omega}})(s,\xi) - \mathcal{F}(u^m)(s,\xi)|d\xi\right)^2 &\leq \left(\int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)||A_{s,\bar\omega}(\xi)|d\xi + \int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)||B_{s,\bar\omega}(\xi)|d\xi\right)^2 \\ &\leq 2(I^1_{s,\bar\omega} + I^2_{s,\bar\omega}), \qquad (7.23)\end{aligned}$$

where

$$\begin{cases} I^1_{s,\bar\omega} := \left(\int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)||A_{s,\bar\omega}(\xi)|d\xi\right)^2 \\ I^2_{s,\bar\omega} := \left(\int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)||B_{s,\bar\omega}(\xi)|d\xi\right)^2, \end{cases} \qquad (7.24)$$

and for all $\bar\omega \in \Omega$, $s \in [0,T]$

$$\begin{cases} A_{s,\bar\omega}(\xi) := \int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} V_s^m(X(\omega))d\eta_{\bar\omega}(\omega) - \int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} V_s^m(X(\omega))dm(\omega) \\ B_{s,\bar\omega}(\xi) := \int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} V_s^{\eta_{\bar\omega}}(X(\omega))d\eta_{\bar\omega}(\omega) - \int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} V_s^m(X(\omega))d\eta_{\bar\omega}(\omega). \end{cases} \qquad (7.25)$$

We observe that $(\bar\omega, s, \xi) \mapsto A_{s,\bar\omega}(\xi)$ and $(\bar\omega, s, \xi) \mapsto B_{s,\bar\omega}(\xi)$ are measurable. Indeed, the map $(\omega, \bar\omega, s, \xi) \mapsto e^{-i\xi\cdot X_s(\omega)} V_s^{\eta_{\bar\omega}}(X(\omega))$ is Borel. By Remark 2.2 we can easily show that for all $N \in \mathbb{N}^\star$, $(\bar\omega, s, \xi) \mapsto 1_{\Delta_N}(\xi)\eta_{\bar\omega}(\omega)$ is (still) a random (finite) measure when Ω is replaced by $\Omega \times [0,T] \times \mathbb{R}^d$ and Δ_N is the centered ball of $\mathbb{R}^d$ with radius N. Proposition 3.3, Chapter 3. of [10] tells us that $(\bar\omega, s, \xi) \mapsto \int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} V_s^{\eta_{\bar\omega}}(X(\omega))1_{\Delta_N}(\xi)d\eta_{\bar\omega}(\omega)$ is measurable, and letting $N \to +\infty$, we observe that $(\bar\omega, s, \xi) \mapsto \int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} V_s^{\eta_{\bar\omega}}(X(\omega))d\eta_{\bar\omega}(\omega)$ is also measurable.

The measurability of A, B follows again by use of Fubini-Tonelli theorem.

Regarding $A_{s,\bar{\omega}}$, let $\varphi_{s,\xi}$ denotes the function defined by $y \in \mathcal{C}^d \mapsto e^{-i\xi\cdot y_s} V_s^m(y)$. Then, one can write $A_{s,\bar{\omega}} = \langle \eta_{\bar{\omega}} - m, \varphi_{s,\xi} \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the pairing between measures and bounded, continuous functionals. $\varphi_{s,\xi}$ is clearly bounded by e^{sM_Λ}; inequalities (2.8) and (3.6) imply the continuity of $\varphi_{s,\xi}$ on $(\mathcal{C}^d, \| \cdot \|_\infty)$, for fixed $(s, \xi) \in \times[0,T] \times \mathbb{R}^d$. By Cauchy-Schwarz inequality we obtain for all $\bar{\omega} \in \Omega$, $s \in [0,T]$

$$
\begin{aligned}
I^1_{s,\bar{\omega}} &\leq \|\mathcal{F}(K)\|_1 \left(\int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)| |A_{s,\bar{\omega}}|^2 d\xi \right) \\
&\leq \|\mathcal{F}(K)\|_1 \left(\int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)| |\langle \eta_{\bar{\omega}} - m, \varphi_{s,\xi} \rangle|^2 d\xi \right) . \qquad (7.26)
\end{aligned}
$$

Since the right-hand side of (7.26) is measurable, taking expectation w.r.t. $d\mathbb{P}(\bar{\omega})$ in both sides yields

$$
\begin{aligned}
\mathbb{E}[I^1_s] &\leq \|\mathcal{F}(K)\|_1 \left(\int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)| \; \mathbb{E}[|\langle \eta - m, \varphi_{s,\xi} \rangle|^2] \; d\xi \right) \\
&\leq e^{2sM_\Lambda} \|\mathcal{F}(K)\|_1 \left(\int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)| \sup_{\substack{\varphi \in \mathcal{C}_b(\mathcal{C}^d) \\ \|\varphi\|_\infty \leq 1}} \mathbb{E}[|\langle \eta - m, \varphi \rangle|^2] d\xi \right) \\
&\leq e^{2sM_\Lambda} \|\mathcal{F}(K)\|_1^2 \sup_{\substack{\varphi \in \mathcal{C}_b(\mathcal{C}^d) \\ \|\varphi\|_\infty \leq 1}} \mathbb{E}[|\langle \eta - m, \varphi \rangle|^2] . \qquad (7.27)
\end{aligned}
$$

Concerning the second term $B_{s,\bar{\omega}}$, for all $(s, \xi) \in [0,T] \times \mathbb{R}^d$, we have

$$
\begin{aligned}
|B_{s,\bar{\omega}}(\xi)|^2 &= \left| \int_{\mathcal{C}^d} e^{-i\xi\cdot X_s(\omega)} \big(V_s^{\eta_{\bar{\omega}}}(X(\omega)) - V_s^m(X(\omega)) \big) d\eta_{\bar{\omega}}(\omega) \right|^2 \\
&\leq \int_{\mathcal{C}^d} \left| V_s^{\eta_{\bar{\omega}}}(X(\omega)) - V_s^m(X(\omega)) \right|^2 d\eta_{\bar{\omega}}(\omega) \\
&\leq e^{2sM_\Lambda} L_\Lambda^2 \int_{\mathcal{C}^d} \left| \int_0^s u^{\eta_{\bar{\omega}}}(r, X_r(\omega)) - u^m(r, X_r(\omega)) dr \right|^2 d\eta_{\bar{\omega}}(\omega) \quad \text{, by (2.8)} \\
&\leq s e^{2sM_\Lambda} L_\Lambda^2 \int_{\mathcal{C}^d} \int_0^s \left| u^{\eta_{\bar{\omega}}}(r, X_r(\omega)) - u^m(r, X_r(\omega)) \right|^2 dr \; d\eta_{\bar{\omega}}(\omega) \\
&\leq s e^{2sM_\Lambda} L_\Lambda^2 \int_0^s \| u^{\eta_{\bar{\omega}}}(r, \cdot) - u^m(r, \cdot) \|_\infty^2 \, dr , \qquad (7.28)
\end{aligned}
$$

where we recall that $\eta_{\bar{\omega}}$ is a probability measure on $\mathcal{C}^d$ for all $\bar{\omega}$ and that functions $(r, x, \bar{\omega}) \in [0,T] \times \mathbb{R}^d \times \Omega \mapsto u^{\eta_{\bar{\omega}}}(r, x)$ and $(r, x) \in [0,T] \times \mathbb{R}^d \mapsto u^m(r, x)$ are uniformly bounded.
Taking into account (7.28), the measurability of the function $(\bar{\omega}, r) \in \Omega \times [0,T] \mapsto \| u^{\eta_{\bar{\omega}}}(r, \cdot) - u^m(r, \cdot) \|_\infty^2$ and Fubini's theorem imply

$$
\begin{aligned}
\mathbb{E}[I^2_s] &\leq \mathbb{E}\left[\left(\int_{\mathbb{R}^d} |\mathcal{F}(K)(\xi)| \sup_{\xi \in \mathbb{R}^d} |B_{s,\cdot}(\xi)| d\xi \right)^2 \right] \\
&\leq \mathbb{E}[\sup_{\xi \in \mathbb{R}^d} |B_{s,\cdot}(\xi)|^2 \; \|\mathcal{F}(K)\|_1^2] \\
&\leq s e^{2sM_\Lambda} L_\Lambda^2 \|\mathcal{F}(K)\|_1^2 \int_0^s \mathbb{E}[\| u^\eta(r, \cdot) - u^m(r, \cdot) \|_\infty^2] dr . \qquad (7.29)
\end{aligned}
$$

Taking the expectation of both sides in (7.23), we inject (7.27) and (7.29) in the expectation of the right-hand side of (7.23) so that (7.21) gives for all $s \in [0,T]$

$$\begin{aligned}
\mathbb{E}\big[\|u^\eta(s,\cdot)-u^m(s,\cdot)\|_\infty^2\big] &\leq C_2(s)\int_0^s \mathbb{E}[\|u^\eta(r,\cdot)-u^m(r,\cdot)\|_\infty^2]dr \\
&\quad + C_1(s)\sup_{\substack{\varphi\in\mathcal{C}_b(\mathcal{C}^d)\\ \|\varphi\|_\infty\leq 1}} \mathbb{E}[|\langle \eta - m, \varphi\rangle|^2], \qquad (7.30)
\end{aligned}$$

where $C_1(s) := \frac{1}{\sqrt{2\pi}} e^{sM_\Lambda}\|\mathcal{F}(K)\|_1^2$ and $C_2(s) := \frac{1}{\sqrt{2\pi}} s e^{2sM_\Lambda} L_\Lambda^2 \|\mathcal{F}(K)\|_1^2$. On one hand, since the functions u^η and u^m are uniformly bounded, $\mathbb{E}[\|u^\eta(s,\cdot)-u^m(s,\cdot)\|_\infty{}^2]$ is finite. On the other hand, observing that $a\mapsto C_1(a)$ and $a \mapsto C_2(a)$ are increasing, we have for all $s \in]0,T], a\in[0,s]$

$$\mathbb{E}\big[\|u^\eta(a,\cdot)-u^m(a,\cdot)\|_\infty^2\big] \leq C_2(s)\int_0^a \mathbb{E}[\|u^\eta(r,\cdot)-u^m(r,\cdot)\|_\infty^2]dr + C_1(s)\sup_{\substack{\varphi\in\mathcal{C}_b(\mathcal{C}^d)\\ \|\varphi\|_\infty\leq 1}} \mathbb{E}[|\langle \eta - m, \varphi\rangle|^2]\,.$$

By Gronwall's lemma, we finally obtain

$$\forall\, s\in[0,T],\ \mathbb{E}\big[\|u^\eta(s,\cdot)-u^m(s,\cdot)\|_\infty^2\big] \leq C_1(s)e^{sC_2(s)}\sup_{\substack{\varphi\in\mathcal{C}_b(\mathcal{C}^d)\\ \|\varphi\|_\infty\leq 1}} \mathbb{E}[|\langle \eta - m, \varphi\rangle|^2]\,. \qquad (7.31)$$

□

7.2 Technicalities about Section 3.2

Lemma 7.1. *We suppose the validity of Assumption 1. Let $r:\ [0,T]\mapsto[0,T]$ be a non-decreasing function such that $r(s)\leq s$ for any $s\in[0,T]$. Let $\mathcal{U}:(t,y)\in[0,T]\times\mathcal{C}^d\to\mathbb{R}$ (respectively $\mathcal{U}':(t,y)\in[0,T]\times\mathcal{C}^d\to\mathbb{R}$), be a given Borel function such that for all $t\in[0,T]$, there is a Borel map $\mathcal{U}_t:\mathcal{C}([0,t],\mathbb{R}^d)\to\mathbb{R}$ (resp. $\mathcal{U}_t^{'}:\mathcal{C}([0,t],\mathbb{R}^d)\to\mathbb{R}$) such that $\mathcal{U}(t,\cdot)=\mathcal{U}_t(\cdot)$ (resp. $\mathcal{U}'(t,\cdot)=\mathcal{U}_t'(\cdot)$). Setting $C_{\Phi,g}(T)=12(4L_\Phi^2+TL_g^2)e^{12T(4L_\Phi^2+TL_g^2)}$, the following two assertions hold.*

1. *Consider Y (resp. Y') a solution of the following SDE for $v=\mathcal{U}$ (resp. $v=\mathcal{U}'$):*

$$Y_t = Y_0 + \int_0^t \Phi(r(s),Y_{r(s)},v(r(s),Y_{\cdot\wedge r(s)}))dW_s + \int_0^t g(r(s),Y_{r(s)},v(r(s),Y_{\cdot\wedge r(s)}))ds\,,\quad \textit{for any } t\in[0,T]\,, \qquad (7.32)$$

where, we emphasize that for all $\theta\in[0,T]$, $Z_{\cdot\wedge\theta}:=\{Z_u, 0\leq u\leq\theta\}\in\mathcal{C}([0,\theta],\mathbb{R}^d)$ for any continuous process Z. For any $a\in[0,T]$, we have

$$\mathbb{E}[\sup_{t\leq a}|Y_t'-Y_t|^2]\leq C_{\Phi,g}(T)\mathbb{E}\left[\int_0^a |\mathcal{U}(r(t),Y_{\cdot\wedge r(t)})-\mathcal{U}'(r(t),Y'_{\cdot\wedge r(t)})|^2dt\right]\,. \qquad (7.33)$$

2. *Suppose moreover that Φ and g are $\frac{1}{2}$-Holder continuous w.r.t. the time variable and Lipschitz w.r.t. the space variables i.e. there exist some positive constants L_Φ and L_g such that for any $(t,t',y,'y',z,z')\in[0,T]^2\times\mathbb{R}^{2d}\times\mathbb{R}^2$*

$$\begin{cases} |\Phi(t,y,z)-\Phi(t',y',z')|\leq L_\Phi(|t-t'|^{\frac{1}{2}}+|y-y'|+|z-z'|)\\ |g(t,y,z)-g(t',y',z')|\leq L_g(|t-t'|^{\frac{1}{2}}+|y-y'|+|z-z'|)\,. \end{cases} \qquad (7.34)$$

Let $r_1,r_2:\ [0,T]\mapsto[0,T]$ being two non-decreasing functions verifying $r_1(s)\leq s$ and $r_2(s)\leq s$ for any $s\in[0,T]$. Let Y (resp. Y') be a solution of (7.32) *for $v=\mathcal{U}$ and $r=r_1$ (resp. $v=\mathcal{U}'$ and $r=r_2$). Then for any $a\in[0,T]$, the following inequality holds:*

$$\begin{aligned}
\mathbb{E}[\sup_{t\leq a}|Y_t'-Y_t|^2] &\leq C_{\Phi,g}(T)\Bigg(\|r_1-r_2\|_1+\int_0^a \mathbb{E}[|Y'_{r_1(t)}-Y'_{r_2(t)}|^2]dt \\
&\quad + \mathbb{E}\left[\int_0^a |\mathcal{U}(r_1(t),Y_{\cdot\wedge r_1(t)})-\mathcal{U}'(r_2(t),Y'_{\cdot\wedge r_2(t)})|^2dt\right]\Bigg)\,, \qquad (7.35)
\end{aligned}$$

where $\|\cdot\|_2$ *is the* $L^2([0,T])$*-norm.*

Proof. 1. Let us consider the first assertion of Lemma 7.1. Let Y (resp. Y') be a solution of (7.32) with associated function $\mathcal{U}$ (resp. $\mathcal{U}'$). Let us fix $a \in]0,T]$. We have

$$Y_\theta - Y'_\theta = \alpha_\theta + \beta_\theta,\ \theta \in [0,a], \tag{7.36}$$

where

$$\begin{aligned}
\alpha_\theta &:= \int_0^\theta \Big(\Phi(r(s), Y_{r(s)}, \mathcal{U}(r(s), Y_{\cdot\wedge r(s)})) - \Phi(r(s), Y'_{r(s)}, \mathcal{U}'(r(s), Y'_{\cdot\wedge r(s)}))\Big)\, dW_s \\
\beta_\theta &:= \int_0^\theta \Big(g(r(s), Y_{r(s)}, \mathcal{U}(r(s), Y_{\cdot\wedge r(s)})) - g(r(s), Y'_{r(s)}, \mathcal{U}'(r(s), Y'_{\cdot\wedge r(s)}))\Big)\, ds\ .
\end{aligned}$$

By BDG inequality, we obtain

$$\begin{aligned}
\mathbb{E}\sup_{\theta\le a}|\alpha_\theta|^2 &\le 4\mathbb{E}\left[\int_0^a \left|\Phi(r(s), Y_{r(s)}, \mathcal{U}(r(s), Y_{\cdot\wedge r(s)})) - \Phi(r(s), Y'_{r(s)}, \mathcal{U}'(r(s), Y'_{\cdot\wedge r(s)}))\right|^2 ds\right] \\
&= 4\int_0^a \mathbb{E}\left[\left|\Phi(r(s), Y_{r(s)}, \mathcal{U}(r(s), Y_{\cdot\wedge r(s)})) - \Phi(r(s), Y'_{r(s)}, \mathcal{U}'(r(s), Y'_{\cdot\wedge r(s)}))\right|^2\right] ds \\
&\le 8L_\Phi^2 \int_0^a \mathbb{E}\left[\left|\mathcal{U}(r(s), Y_{\cdot\wedge r(s)}) - \mathcal{U}'(r(s), Y'_{\cdot\wedge r(s)})\right|^2\right] ds + 8L_\Phi^2 \int_0^a \mathbb{E}\left[|Y_{r(s)} - Y'_{r(s)}|^2\right] ds\ .
\end{aligned} \tag{7.37}$$

Concerning β in (7.36), by Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\mathbb{E}\sup_{\theta\le a}|\beta_\theta|^2 &\le a\mathbb{E}\left[\int_0^a |g(r(s), Y_{r(s)}, \mathcal{U}(r(s), Y_{\cdot\wedge r(s)})) - g(r(s), Y'_{r(s)}, \mathcal{U}'(r(s), Y'_{\cdot\wedge r(s)}))|^2 ds\right] \\
&\le 2aL_g^2\mathbb{E}\left[\int_0^a |\mathcal{U}(r(s), Y_{\cdot\wedge r(s)}) - \mathcal{U}'(r(s), Y'_{\cdot\wedge r(s)})|^2 ds\right] + 2aL_g^2\int_0^a \mathbb{E}\left[|Y_{r(s)} - Y'_{r(s)}|^2\right] ds\ .
\end{aligned} \tag{7.38}$$

Gathering (7.38) together with (7.37) and using the fact that $r(s) \le s$, implies

$$\begin{aligned}
\mathbb{E}[\sup_{\theta\le a}|Y'_\theta - Y_\theta|^2] &\le 4(4L_\Phi^2 + TL_g^2)\bigg(\mathbb{E}[\int_0^a |\mathcal{U}(r(s), Y_{\cdot\wedge r(s)}) - \mathcal{U}'(r(s), Y'_{\cdot\wedge r(s)})|^2 ds] \\
&\quad + \int_0^a \mathbb{E}[|Y_{r(s)} - Y'_{r(s)}|^2]\, ds\bigg) \\
&\le 4(4L_\Phi^2 + TL_g^2)\bigg(\mathbb{E}[\int_0^a |\mathcal{U}(r(s), Y_{\cdot\wedge r(s)}) - \mathcal{U}'(r(s), Y'_{\cdot\wedge r(s)})|^2 ds] \\
&\quad + \int_0^a \mathbb{E}[\sup_{\theta\le s}|Y_\theta - Y'_\theta|^2]\, ds\bigg)\ ,
\end{aligned}$$

for any $a \in [0,t]$.

We conclude the proof by applying Gronwall's lemma.

2. Consider now the second assertion of Lemma 7.1. Following the same lines as the proof of assertion 1. and using the Lipschitz property of Φ and g w.r.t. to both the time and space variables (7.34), we

obtain the inequality

$$
\begin{aligned}
\mathbb{E}[\sup_{t\le a}|Y'_t - Y_t|^2] &\le 12(4L_\Phi^2 + TL_g^2)\Big(\int_0^a |r_1(t)-r_2(t)|dt + \int_0^a \mathbb{E}[|Y'_{r_1(t)} - Y'_{r_2(t)}|^2]dt \\
&\quad + \mathbb{E}[\int_0^a |\mathcal{U}(r_1(t), Y_{\cdot\wedge r_1(t)}) - \mathcal{U}'(r_2(t), Y'_{\cdot\wedge r_2(t)})|^2 dt] + \int_0^a \mathbb{E}[|Y_{r_1(t)} - Y'_{r_1(t)}|^2]\, dt\Big) \\
&\le 12(4L_\Phi^2 + TL_g^2)\Big(\|r_1 - r_2\|_1 + \int_0^a \mathbb{E}[|Y'_{r_1(t)} - Y'_{r_2(t)}|^2]dt \\
&\quad + \mathbb{E}[\int_0^a |\mathcal{U}(r_1(t), Y_{\cdot\wedge r_1(t)}) - \mathcal{U}'(r_2(t), Y'_{\cdot\wedge r_2(t)})|^2 dt] + \int_0^a \mathbb{E}[\sup_{s\le t}|Y_s - Y'_s|^2]\, dt\Big) .
\end{aligned}
$$

Applying again Gronwall's lemma concludes the proof.

□

7.3 Some technical proofs of the convergence of approximating sequences related to Section 4 and Section 5

The results stated and established in this section are the main tools that will be used in Section 4 and Section 5 to prove the main theorems, i.e. Theorem 4.2 and Theorem 5.1.

Lemma 7.2. *Let $(\mathbb{P}_n)_{n\ge 0}$ be a sequence of probability measures on $\mathcal{C}^d$ converging weakly to some probability $\mathbb{P}$. Let $(f_n)_{n\ge 0}$ be a uniformly bounded sequence of real-valued, continuous functions defined on $\mathcal{C}^d$, converging uniformly on every compact subset to some continuous f. Then $\int_{\mathcal{C}^d} f_n(\omega)d\mathbb{P}_n(\omega) \xrightarrow[n\to+\infty]{} \int_{\mathcal{C}^d} f(\omega)d\mathbb{P}(\omega)$.*

Remark 7.3. *We apply several times Lemma 7.2. We will verify its assumptions showing that the sequence (f_n) converges uniformly on each bounded ball of $\mathcal{C}^d$. This will be enough since every compact of $\mathcal{C}^d$ is bounded.*

We emphasize that the hypothesis of uniform convergence in Lemma 7.2 is crucial, see remark below,

Remark 7.4. *Let define $\Omega = [0,1]$ equipped with the Borel σ-field, $(Z_n)_{n\ge 0}$ a sequence of continuous, real-valued functions s.th.*

$$
\begin{cases} 0 & , \quad x \ge \frac{2}{n} \\ nx & , \quad x\in[0,\frac{1}{n}] \\ -nx+2 & , \quad x\in[\frac{1}{n},\frac{2}{n}]. \end{cases} \tag{7.39}
$$

We consider a sequence of probability measures $(m_n)_{n\ge 0}$ s.th. $m_n(dx) = \delta_{\frac{1}{n}}(dx)$ and $m_0(dx) = \delta_0(dx)$.

On one hand, we can observe the following.

- $Z_n \xrightarrow[n\to+\infty]{} 0$*, pointwise.*
- *for all $n\ge 0$, $|Z_n|\le 1$, surely.*
- $m_n \xrightarrow[n\to+\infty]{} m$*, weakly.*

On the other hand,$\int_0^1 Z_n dm_n = Z_n(\frac{1}{n}) = 1 \nrightarrow 0$.

The proof of Proposition 4.3 goes through several steps.

We first formulate below an useful elementary result, which follows by a simple application of Lebesgue dominated convergence theorem. It will be often used in the sequel.

Lemma 7.5. *Let $\Lambda : [0,T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel bounded function such that for almost all $t \in [0,T]$ $\Lambda(t,\cdot,\cdot)$ is continuous. The function $F : [0,T] \times \mathcal{C}^d \times \mathcal{C} \longrightarrow \mathbb{R}$, $x_0 \in \mathbb{R}$, defined by $F(t,y,z) = K(x_0 - y_t) \exp\left(\int_0^t \Lambda(r,y_r,z_r)dr\right)$ is continuous.*

The proposition below establishes an important result about the convergence of the sequence $(u_n)_{n\in\mathbb{N}}$.

Proposition 7.6. *Let $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of Borel uniformly bounded functions defined on $[0,T] \times \mathbb{R}^d \times \mathbb{R}$, such that for every n, $\Lambda_n(t,\cdot,\cdot)$ is continuous. Assume the existence of a Borel function $\Lambda : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ such that, for almost all $t \in [0,T]$, $\left[\Lambda_n(t,.,.) - \Lambda(t,.,.)\right] \xrightarrow[n \to +\infty]{} 0$, uniformly on each compact of $\mathbb{R}^d \times \mathbb{R}$.*
Let $(Y^n)_{n\in\mathbb{N}}$ be a sequence of continuous processes. We consider a sequence (u_n) such that, for all $(t,x) \in [0,T] \times \mathbb{R}^d$

$$\begin{cases} u_n(t,x) = \int_{\mathcal{C}^d} K(x - X_t(\omega)) \exp\left\{\int_0^t \Lambda_n\big(r, X_r(\omega), u_n(r, X_r(\omega))\big)dr\right\} dm^n(\omega) \\ m^n := \mathcal{L}(Y^n) . \end{cases} \tag{7.40}$$

We set $Z^n := u^n(\cdot, Y^n)$ for all $n \in \mathbb{N}$. Suppose moreover that $\nu^n := \mathcal{L}(Y^n, Z^n)$ converges weakly to some Borel probability measures ν on $\mathcal{C}^d \times \mathcal{C}$.
Then, (u_n) converges uniformly on each compact of $[0,T] \times \mathbb{R}^d$ to a function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$u(t,x) = \int_{\mathcal{C}^d \times \mathcal{C}} K(x - X_t(\omega)) \exp\left\{\int_0^t \Lambda\big(r, X_r(\omega), X'_r(\omega')\big)dr\right\} d\nu(\omega,\omega') , \tag{7.41}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$. In particular u is continuous.

The proof of Proposition 7.6 is based on a technical lemma.

Lemma 7.7. *Let (Λ_n), Λ be as stated in Proposition 7.6. Let $x_0 \in \mathbb{R}^d$, we designate by $F_n, F : [0,T] \times \mathcal{C}^d \times \mathcal{C} \to \mathbb{R}$, the maps*

$$F_n(t,y,z) := K(x_0 - y_t) \exp\left(\int_0^t \Lambda_n(r,y_r,z_r)\right) \quad \textit{and} \quad F(t,y,z) := K(x_0 - y_t) \exp\left(\int_0^t \Lambda(r,y_r,z_r)\right).$$

Then for every $M > 0$, F_n converges to F when n goes to infinity uniformly with respect to $(t,y,z) \in [0,T] \times B_d(0,M) \times B_1(0,M)$, with $B_k(O,M) := \{y \in \mathcal{C}^k, ||y||_\infty := \sup_{u\in[0,T]} |y_u| \leq M\}$ for $k \in \mathbb{N}^\star$.

Proof of Lemma 7.7. We want to evaluate $||F_n - F||_{\infty,M} := \sup_{(t,y,z)\in[0,T]\times B_d(O,M)\times B_1(0,M)} |F_n(t,y,z) - F(t,y,z)|$.

Since $(\Lambda_n)_{n\geq 0}$ are uniformly bounded, there is a constant M_Λ such that

$$\forall r \in [0,T], \quad \sup_{(y',z')\in B_d(O,M)\times B_1(0,M)} |\Lambda_n(r,y'_r,z'_r) - \Lambda(r,y'_r,z'_r)| \leq 2M_\Lambda.$$

By use of (2.7), we obtain for all $(t,y,z) \in [0,T] \times B_d(O,M) \times B_1(0,M)$,

$$|F_n(t,y,z) - F(t,y,z)| \leq M_K \exp(M_\Lambda) \int_0^t \sup_{(y',z')\in B_d(O,M)\times B_1(0,M)} |\Lambda_n(r,y'_r,z'_r) - \Lambda(r,y'_r,z'_r)|dr, \tag{7.42}$$

which implies

$$||F_n - F||_{\infty,M} \leq M_K \exp(M_\Lambda) \int_0^T \sup_{(y',z')\in B_d(O,M)\times B_1(0,M)} |\Lambda_n(r,y'_r,z'_r) - \Lambda(r,y'_r,z'_r)|dr. \tag{7.43}$$

By Lebesgue's dominated convergence theorem, we have

$$\int_0^T \sup_{(y',z')\in B_d(O,M)\times B_1(0,M)} |\Lambda_n(r,y'_r,z'_r) - \Lambda(r,y'_r,z'_r)| \, dr \longrightarrow 0 ,$$

which concludes the proof of Lemma 7.7. □

Now, we proceed with the proof of Proposition 7.6.

Proof of Proposition 7.6. The first step consists in proving the pointwise convergence of $(u_n)_{n\in\mathbb{N}}$. Observe that $u_n(t,x) = \int_{\mathcal{C}^d\times\mathcal{C}} K(x - X_t(\omega)) \exp\left\{\int_0^t \Lambda_n\big(r, X_r(\omega), X'_r(\omega')\big)dr\right\} d\nu^n(\omega,\omega')$. Let us fix $t \in [0,T], x \in \mathbb{R}^d$. Let us introduce the sequence of real valued functions $(f_n)_{n\in\mathbb{N}}$ and f defined on $\mathcal{C}^d \times \mathcal{C}$ such that

$$f_n(y,z) = K(x-y_t) \exp\left\{\int_0^t \Lambda_n\big(r,y_r,z_r\big)dr\right\} \quad \text{and} \quad f(y,z) = K(x-y_t) \exp\left\{\int_0^t \Lambda\big(r,y_r,z_r\big)dr\right\} .$$

By Lemma 7.5, f_n and f are continuous.
By Lemma 7.7, it follows that $f_n \xrightarrow[n \longrightarrow +\infty]{} f$ uniformly on each closed ball (and therefore also for each compact subset) of $\mathcal{C}^d \times \mathcal{C}$. Then applying Lemma 7.2 and Remark 7.3, with $\mathcal{C}^d \times \mathcal{C}$, $\mathbb{P} = \nu$, $\mathbb{P}^n = \nu^n$ allows to conclude that $(u_n)_{n\in\mathbb{N}}$ converges pointwise to u when n goes to ∞, with u defined by (7.41).

We go on proving the uniform convergence of $(u^n)_{n\in\mathbb{N}}$ on each compact of $[0,T]\times\mathbb{R}^d$.

We fix a compact C of $\mathbb{R}^d$. The restrictions of u_n to $[0,T]\times C$ are uniformly bounded. Provided we prove that the sequence $(u_n|_{[0,T]\times C})$ is equicontinuous, Ascoli-Arzela theorem would imply that the set of restrictions of u_n to $[0,T]\times C$ is relatively compact with respect to uniform convergence norm topology.
To conclude, given a subsequence (u_{n_k}) it is enough to extract a subsubsequence converging to u. Since the set of restrictions of u_{n_k} to C is relatively compact, there is a function $v : [0,T]\times C \to \mathbb{R}$ to which u_{n_k} converges uniformly on $[0,T]\times C$. Since (u_n) converges pointwise to u, obviously v coincides with u on $[0,T]\times C$.
It remains to show the equicontinuity of the sequence (u_n) on $[0,T]\times C$. We do this below.
Let $\varepsilon' > 0$. We need to prove that $\exists \delta, \eta > 0, \forall (t,x),(t',x') \in [0,T]\times C$,

$$|t-t'| < \delta,\ |x-x'| < \eta \Longrightarrow \forall n \in \mathbb{N},\ |u_n(t,x) - u_n(t',x')| < \varepsilon'. \tag{7.44}$$

We start decomposing as follows:

$$|u_n(t,x) - u_n(t',x')| \le |(u_n(t,x) - u_n(t,x'))| + |(u_n(t,x') - u_n(t',x'))|. \tag{7.45}$$

As far as the first term in the right-hand side of (7.45) is concerned, we have

$$\begin{aligned} |u_n(t,x) - u_n(t,x')| &\le \int_{\mathcal{C}^d} |K(x - X_t(\omega)) - K(x' - X_t(\omega))| \exp(M_\Lambda T) dm^n(\omega), \\ &\le \exp(M_\Lambda T) L_K |x-x'|, \end{aligned} \tag{7.46}$$

where the constant M_Λ is an uniform upper bound of $(|\Lambda_n|, n \ge 0)$. We choose $\eta = \frac{\varepsilon'}{3\exp(M_\Lambda T)L_K}$ to obtain

$$|(u_n(t,x) - u_n(t,x'))| \le \frac{\varepsilon'}{3}, \tag{7.47}$$

for $x, x' \in C$ such that $|x-x'| < \eta$ and $t \in [0,T]$.
Regarding the second one we have

$$|u_n(t,x') - u_n(t',x')| \le B_1 + B_2, \tag{7.48}$$

where

$$
\begin{array}{rcl}
B_1 & := & \Big|\int_{\mathcal{C}^d} \left[K(x'-X_t(\omega)) - K(x'-X_{t'}(\omega))\right] \exp\left\{\int_0^t \Lambda_n\big(r, X_r(\omega), u_n(r, X_r(\omega))\big)dr\right\} dm^n((\omega))\Big| \\
B_2 & := & \Big|\int_{\mathcal{C}^d} K(x'-X_{t'}(\omega))\left[\exp\left\{\int_0^t \Lambda_n\big(r, X_r(\omega), u_n(r, X_r(\omega))\big)dr\right\} - \right. \\
 & & \left. \exp\left\{\int_0^{t'} \Lambda_n\big(r, X_r(\omega), u_n(r, X_r(\omega))\big)dr\right\}\right] dm^n(\omega)\Big| .
\end{array}
\tag{7.49}
$$

We first estimate B_1. We fix $\varepsilon > 0$. Since (m_n) are tight, by Proposition 7.9 there is $\delta_\varepsilon > 0$ such that

$$
\forall n \in \mathbb{N},\ \mathbb{P}(\Omega^n_{\varepsilon,\delta_\varepsilon}) \le \varepsilon\ , \tag{7.50}
$$

where $\Omega^n_{\varepsilon,\delta_\varepsilon} := \left\{\bar\omega \in \Omega\ \Big|\ \sup_{(t,t')\in[0,T]^2, |t-t'|\le\delta_\varepsilon} |Y^n_t(\bar\omega) - Y^n_{t'}(\bar\omega)| \ge \varepsilon\right\}$.
In the sequel of the proof, for simplicity we will simply write $\Omega^n_\varepsilon := \Omega^n_{\varepsilon,\delta_\varepsilon}$. Suppose that $|t-t'| \le \delta_\varepsilon$. Then, for all $x' \in C$

$$
\begin{array}{rcl}
B_1 & = & \Big|\mathbb{E}\Big[\Big(K(x'-Y^n_t) - K(x'-Y^n_{t'})\Big)\exp\Big\{\displaystyle\int_0^t \Lambda(r, Y^n_r, u^n(r, Y^n_r))\Big\}\Big]\Big| \\
 & \le & \exp(M_\Lambda T)\mathbb{E}\Big[|K(x'-Y^n_t) - K(x'-Y^n_{t'})|\Big],
\end{array}
$$

where

$$
\begin{array}{rcl}
I_1(\varepsilon, n) & := & \mathbb{E}\Big[|K(x'-Y^n_t) - K(x'-Y^n_{t'}|\ 1_{\Omega^n_\varepsilon}\Big]
\end{array}
\tag{7.51}
$$

$$
\begin{array}{rcl}
I_2(\varepsilon, n) & := & \mathbb{E}\Big[|K(x'-Y^n_t) - K(x'-Y^n_{t'}|\ 1_{(\Omega^n_\varepsilon)^c}\Big]\ .
\end{array}
\tag{7.52}
$$

We have

$$
I_1(\varepsilon, n) \le 2M_K\mathbb{P}(\Omega^n_\varepsilon) \le 2M_K\varepsilon\ , \tag{7.53}
$$

and

$$
I_2(\varepsilon, n) \le L_K\mathbb{E}\Big[|Y^n_t - Y^n_{t'}|\ 1_{(\Omega^n_\varepsilon)^c}\Big] \le \varepsilon L_K. \tag{7.54}
$$

At this point, we have shown that for $|t-t'| \le \delta_\varepsilon$, $x' \in C$,

$$
B_1 \le \varepsilon(2M_K + L_K)\exp(M_\Lambda T)\ . \tag{7.55}
$$

We can now choose $\varepsilon := \frac{\varepsilon'}{3(2M_K+L_K)}\exp(-M_\Lambda T)$ so that $B_1 \le \frac{\varepsilon'}{3}$.
Concerning the term B_2, using (2.7), we have

$$
\begin{array}{rcl}
B_2 & \le & \int_{\mathcal{C}^d} |K(x'-X_{t'}(\omega))|\Big| e^{\left\{\int_0^t \Lambda_n\big(r, X_r(\omega), u_n(r, X_r(\omega))\big)dr\right\}} - e^{\left\{\int_0^{t'} \Lambda_n\big(r, X_r(\omega), u_n(r, X_r(\omega))\big)dr\right\}}\Big| dm^n(\omega) \\
 & \le & M_K\exp(M_\Lambda T)\int_{\mathcal{C}^d} dm^n(\omega)\left|\int_t^{t'} \Lambda_n\big(r, X_r(\omega), u_n(r, X_r(\omega))\big)dr\right| \\
 & \le & M_K\exp(M_\Lambda T)M_\Lambda|t-t'|\ .
\end{array}
\tag{7.56}
$$

We choose $\delta = \min(\delta_\epsilon, \frac{\varepsilon'}{3M_K M_\Lambda \exp(M_\Lambda)})$. For $|t-t'| < \delta$, we have $B_2 \le \frac{\varepsilon'}{3}$. By additivity $B_1 + B_2 \le \frac{2\varepsilon'}{3}$ and finally, taking into account (7.47) and (7.48), (7.44) is verified. This concludes the proof of Proposition 7.6. □

We end this section by recalling the following classical result on strong convergence of solutions of SDEs.

Lemma 7.8. *Let R_0 be a square integrable random variable on some filtered probability space, equipped with a p dimensional Brownian motion W. Let $a_n : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^{d\times p}$ and $b_n : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ Borel functions verifying the following.*

- $\exists L > 0$, *for all* $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, $\sup_{n \geq 0} |a_n(t, x) - a_n(t, y)| + \sup_{n \geq 0} |b_n(t, x) - b_n(t, y)| \leq L|x - y|$;
- $\exists c > 0$, *for all* $x \in \mathbb{R}^d$, $\sup_{n \geq 0}(|a_n(t, x)| + |b_n(t, x)|) \leq c(1 + |x|)$;
- $(a_n), (b_n)$ *converge pointwise respectively to Borel functions* $a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times p}$ *and* $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$.

Then there exists a unique strong solution of

$$\begin{cases} dY_t = a(t, Y_t)dW_t + b(t, Y_t)dt \\ Y_0 = R_0. \end{cases} \tag{7.57}$$

Moreover, let for each n, *let the strong solution* X^n *(which of course exists) of*

$$\begin{cases} dY_t^n = a_n(t, Y_t^n)dW_t + b_n(t, Y_t^n)dt \\ Y_0^n = R_0. \end{cases} \tag{7.58}$$

Then,

$$\sup_{t \leq T} |Y_t^n - Y_t| \xrightarrow[n \longrightarrow +\infty]{L^2} 0.$$

Proof. The existence and uniqueness of Y follows because a, b are Lipschitz with linear growth.
The proof of the convergence is classical: it relies on BDG and Jensen's inequalities together with Gronwall's lemma. $\square$

7.4 Tightness of the approximating sequences of processes related to Sections 4 and 5.

Before stating a tightness criterion for our family of approximating sequences we need to express the classical Theorem of Kolmogorov-Centsov, stated in Theorem 4.10, Chapter 2 in [12], taking into account Remark 4.13.

Proposition 7.9. *Let* $r \in \mathbb{N}^\star$. *A sequence* $(\mathbb{P}_n)_{n \geq 0}$ *of Borel probability measures on* $\mathcal{C}^r$ *is tight if and only if*

- $$\lim_{\lambda \longrightarrow +\infty} \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\{ \omega \in \mathcal{C}^r \mid |\omega_0| > \lambda \} \right) = 0 \, , \tag{7.59}$$
- $\forall (\varepsilon, s, t) \in \mathbb{R}_+^\star \times [0, T] \times [0, T]$,
$$\lim_{\delta \downarrow 0} \sup_{n \in \mathbb{N}} \mathbb{P}_n (\{ \omega \in \mathcal{C}^r \mid \max_{\substack{(s,t) \in [0,T]^2 \\ |t-s| \leq \delta}} |\omega_t - \omega_s| > \varepsilon \}) = 0 \, . \tag{7.60}$$

The following tightness result will be used in the proofs of Theorems 4.2 and 5.1.

Lemma 7.10. *Let* $K : \mathbb{R}^d \to \mathbb{R}$ *be bounded and Lipschitz. For* $n \in \mathbb{N}$, *let* $\Lambda_n : [0, T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ *be uniformly bounded in* n. *We consider Borel functions* $\Phi_n : [0, T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^{d \times p}$, $g_n : [0, T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^d$, *which have linear growth with respect to* (y, z) *(uniformly with respect to time* t*), with rates uniformly bounded in* n. *We also consider a tight sequence* (ζ_0^n) *of probability measures on* $\mathbb{R}^d$. *Let* (Y^n, u_n) *be solutions of*

$$\begin{cases} dY_t^n = \Phi_n(t, Y_t^n, u_n(t, Y_t^n))dW_t + g_n(t, Y_t^n, u_n(t, Y_t^n))dt \\ u_n(t, x) := \int_{\mathcal{C}^d} K(x - X_t(\omega)) \, \exp\left\{ \int_0^t \Lambda_n\big(r, X_r(\omega), u_n(r, X_r(\omega))\big)dr \right\} dm^n(\omega) \\ m_n = \mathcal{L}(Y_n), \end{cases} \tag{7.61}$$

where for all $n \in \mathbb{N}$, Y_0^n *is a r.v. distributed according to* ζ_0^n.
Then, the family $\big(\nu^n = \mathcal{L}(Y_\cdot^n, u_n(\cdot, Y_\cdot^n)), \ n \geq 0\big)$ *is tight.*

Remark 7.11. *In the proof below we will make use of the following classical statement, which can be established easily with the help BDG and Cauchy-Schwarz inequalities.*
Let $\tilde{\Phi} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times p}$ and $\tilde{g} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be Borel functions with linear growth in the second variable (uniformly with respect to the time variable). Let Y be solution of the SDE

$$Y_t = Y_0 + \int_0^t \tilde{\Phi}(s, Y_s) dW_s + \int_0^t \tilde{g}(s, Y_s) ds,$$

where W is a p-dimensional Brownian motion. Then, for every $k \geq 1$, there is a constant C only depending on the linear growth rate and constant of $\tilde{\Phi}$ and $\tilde{g}$ such that

$$\mathbb{E}\left((Y_t - Y_s)^{2k}\right) \leq C|t-s|^k, \forall s, t \in [0,T].$$

Proof. If we indicate by $\mathbb{P}_n$ the law of $(Y_n, u^n(\cdot, Y^n))$ we bound the l.h.s. of (7.59) as follows:

$$\begin{aligned}
\mathbb{P}_n(\{\omega \in \mathcal{C}^{d+1} | \, |\omega_0| > \lambda\}) &= \mathbb{P}(\{|(Y_0^n, u^n(0, Y_0^n))| > \lambda\}) \\
&\leq \mathbb{P}(\{|Y_0^n| + |u^n(0, Y_0^n)| > \lambda\}) \\
&\leq \mathbb{P}(\{|Y_0^n| > \frac{\lambda}{2}\}) + \mathbb{P}(\{|u^n(0, Y_0^n)| > \frac{\lambda}{2}\}) \\
&\leq \zeta_0^n(\{x \in \mathbb{R}^d | \, |x| > \frac{\lambda}{2}\}) + \mathbb{P}(\{|u^n(0, Y_0^n)| > \frac{\lambda}{2}\}). \qquad (7.62)
\end{aligned}$$

Let us fix $\varepsilon > 0$. On one hand, (ζ_0^n) being tight there exists a compact set $\mathfrak{K}_\varepsilon$ of $\mathbb{R}^d$ such that $\sup_{n\in\mathbb{N}} \zeta_0^n(\mathfrak{K}_\varepsilon^c) \leq \varepsilon$. Then, there exists $\lambda_\varepsilon > 0$ such that $\{x \in \mathbb{R}^d | \, |x| > \frac{\lambda_\varepsilon}{2}\} \subset \mathfrak{K}_\varepsilon^c$ which implies

$$\sup_{n\in\mathbb{N}} \zeta_0^n(\{x \in \mathbb{R}^d | \, |x| > \frac{\lambda_\varepsilon}{2}\}) \leq \sup_{n\in\mathbb{N}} \zeta_0^n(\mathfrak{K}_\varepsilon^c) \leq \varepsilon.$$

On the other hand, by item 2. of Remark 3.2, $|u^n|$ is uniformly bounded by $M_K \exp(M_\Lambda T)$. Consequently, for all $\lambda > 2M_K \exp(M_\Lambda T)$,

$$\mathbb{P}(\{|u^n(0, Y_0^n)| > \frac{\lambda}{2}\}) = 0. \qquad (7.63)$$

Consequently for $\lambda \geq \max(\lambda_\varepsilon, M_K \exp(M_\Lambda T))$, we get

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n(\{\omega \in \mathcal{C}^{d+1} | \, |\omega_0| > \lambda\}) \leq \varepsilon \, . \qquad (7.64)$$

Taking the limit when λ goes to infinity, we finally get inequality (7.59) since $\varepsilon > 0$ is arbitrary.

It remains to prove (7.60).

We will make use of Garsia-Rodemich-Rumsey Theorem, see e.g. Theorem 2.1.3, Chapter 2 in [17] or [3]. We will show that, for all $0 \leq s < t \leq T$, there exists a positive real constant $C \geq 0$

$$\mathbb{E}[|Y_t^n - Y_s^n|^4 + |u_n(t, Y_t^n) - u_n(s, Y_s^n)|^4] \leq C|t-s|^2 \, , \qquad (7.65)$$

where C does not depend on n. Suppose for a moment that (7.65) holds true.
Let $\varepsilon > 0$ fixed. Let $\delta > 0$. If $\mathbb{P}_n$ denotes again the law of $(Y^n, u^n(\cdot, Y^n))$, the quantity

$$\mathbb{P}_n(\{\omega \in \mathcal{C}^{d+1} \mid \sup_{\substack{(s,t)\in[0,T]^2 \\ |t-s|\leq\delta}} |\omega_t - \omega_s| > \varepsilon\}) \qquad (7.66)$$

intervening in (7.60) is bounded, up to a constant, by

$$\mathbb{P}(\max_{\substack{(s,t)\in[0,T]^2 \\ |t-s|\leq\delta}} \{|Y_t^n - Y_s^n| + |u^n(t, Y_t^n) - u^n(s, Y_s^n)|\} > \varepsilon) \, . \qquad (7.67)$$

Let us fix $\gamma \in]0, \frac{1}{4}[$. By Garsia-Rodemich-Rumsey theorem, there is a sequence of non-negative r.v. Γ^n such that, a.s.

$$\sup_{n\in\mathbb{N}} \mathbb{E}[(\Gamma^n)^4] < \infty$$

$$\forall (s,t) \in [0,T]^2,\ |Y_t^n - Y_s^n| + |u^n(t, Y_t^n) - u^n(s, Y_s^n)| \le \Gamma^n |t-s|^\gamma. \tag{7.68}$$

If $|t-s| \le \delta$ (7.68) gives

$$\max_{\substack{(s,t)\in[0,T]^2\\ |t-s|\le\delta}} \{|Y_t^n - Y_s^n| + |u^n(t, Y_t^n) - u^n(s, Y_s^n)|\} \le \Gamma^n \delta^\gamma \ . \tag{7.69}$$

By (7.69) and Chebyshev's inequality, for any $n \in \mathbb{N}$, the quantity (7.66) is bounded by

$$\begin{aligned} \mathbb{P}(\Gamma^n \delta^\gamma > \varepsilon) &= \mathbb{P}(\Gamma^n > \varepsilon \delta^{-\gamma}) \\ &\le \frac{\delta^{4\gamma}}{\varepsilon^4}, \end{aligned}$$

for any $n \in \mathbb{N}$. Since $\delta > 0$ is arbitrary, (7.60) follows. To conclude the proof of the lemma, it remains to show (7.65). For this, let $0 \le s < t \le T$. We have first to bound $\mathbb{E}[|Y_t^n - Y_s^n|^4]$.

We set $\tilde{\Phi}_n(t,y) := \Phi_n(t,y,u_n(t,y)), \tilde{g}_n(t,y) := g_n(t,y,u_n(t,y))$. Let $\tilde{L}_\Phi$ (resp. $\tilde{L}_g$) denote the uniform linear growth rate of Φ_n (resp. g_n), by item 2. in Remark 3.2, we see that $\tilde{\Phi}_n$ (resp. $\tilde{g}_n$) have uniform linear growth with rate $\tilde{L}_\Phi(1+\max(L_K, M_K)e^{M_\Lambda T})$ (resp. $\tilde{L}_g(1+\max(L_K, M_K)e^{M_\Lambda T}))$. Consequently by Remark 7.11, for every $k \ge 1$,

$$\mathbb{E}[|Y_t^n - Y_s^n|^{2k}] \le C'|t-s|^k, \tag{7.70}$$

where the constant C' does not depend on n and only depends on $\tilde{L}_\Phi, \tilde{L}_g, m_\Phi, m_g, M_\Lambda, L_K, k, T$.
Regarding the second expectation in (7.65), we get

$$\begin{aligned} \mathbb{E}[|u_n(t, Y_t^n) - u_n(s, Y_s^n)|^4] &= \int_{\mathcal{C}^d} \big(u_n(t, X_t(\omega)) - u_n(s, X_s(\omega))\big)^4 dm^n(\omega) \\ &\le 8(I_1 + I_2)\ , \end{aligned} \tag{7.71}$$

where

$$\begin{aligned} I_1 &:= \int_{\mathcal{C}^d} \big(u_n(t, X_t(\omega)) - u_n(s, X_t(\omega))\big)^4 dm^n(\omega) \\ I_2 &:= \int_{\mathcal{C}^d} \big(u_n(s, X_t(\omega)) - u_n(s, X_s(\omega))\big)^4 dm^n(\omega)\ . \end{aligned} \tag{7.72}$$

On one hand, for all $x \in \mathbb{R}^d$,

$$\begin{aligned} |u_n(t,x) - u_n(s,x)| &= \Big|\mathbb{E}\Big[K(x - Y_t^n)e^{\int_0^t \Lambda_n(r, Y_r^n, u_n(r, Y_r^n))dr}\Big] - \mathbb{E}\Big[K(x - Y_s^n)e^{\int_0^s \Lambda_n(r, Y_r^n, u_n(r, Y_r^n))dr}\Big]\Big| \\ &\le \int_{\mathcal{C}^d} |K(x - X_t(\omega)) - K(x - X_s(\omega))| \exp\left(\int_0^t \Lambda_n(r, X_r, u_n(r, X_r))dr\right) dm^n(\omega) \\ &\quad + \int_{\mathcal{C}^d} K(x - X_s(\omega)) \left| \exp\left(\int_0^t \Lambda_n(r, X_r(\omega), u_n(r, X_r(\omega))\right) dr \right. \\ &\quad \left. - \exp\left(\int_0^s \Lambda_n(r, X_r(\omega), u_n(r, X_r(\omega)))dr\right) \right| dm^n(\omega) \end{aligned}$$

By (2.7) and (7.70) (with $k = 1$) together with Cauchy-Schwarz inequality, this is lower than

$$\begin{aligned} & L_K \exp(M_\Lambda T) \int_{\mathcal{C}^d} |X_t(\omega) - X_s(\omega)| dm^n(\omega) \\ + \quad & M_K \exp(M_\Lambda T) \int_{\mathcal{C}^d} \Big| \int_s^t \Lambda_n(r, X_r(\omega), u_n(r, X_r(\omega))) dr \Big| dm^n(\omega) \\ \leq \quad & (L_K \exp(M_\Lambda T)\sqrt{C'} + M_K \exp(M_\Lambda T) M_\Lambda \sqrt{T}) \sqrt{|t-s|}, \end{aligned}$$

where M_K is an upper bound of K. This implies

$$I_1 = \int_{\mathcal{C}^d} |u_n(t, X_t(\omega)) - u_n(s, X_t(\omega))|^4 dm^n(\omega) \quad \leq \quad (L_K \exp(M_\Lambda T)\sqrt{C'} + M_K \exp(M_\Lambda T) M_\Lambda \sqrt{T})^4 |t-s|^2 . \tag{7.73}$$

(3.2) in item 2. of Remark 3.2 implies

$$\begin{aligned} I_2 = \int_{\mathcal{C}^d} |u_n(s, X_t(\omega)) - u_n(s, X_s(\omega))|^4 dm^n(\omega) \quad & \leq \quad L_K \exp(M_\Lambda T) \int_{\mathcal{C}^d} |X_t(\omega) - X_s(\omega)|^4 dm^n(\omega) \\ & \leq \quad L_K \exp(M_\Lambda T) C' |t-s|^2 , \end{aligned} \tag{7.74}$$

where the second inequality comes from (7.70) with $k = 2$.
Coming back to (7.71), we have $|I_1 + I_2| \leq C''|t-s|^2$ with C'' a constant value depending only on $T, \tilde{L}_\Phi, \tilde{L}_g, m_\Phi, m_g, M_\Lambda, M_K, L_K, T$. This enable us to conclude the proof of (7.65) and finally the one of Lemma 7.10.

□

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