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Abstract

This work deals with an optimization problem for three-stage space launcher. The mission of the launcher is to put a given payload on the GEO orbit with the minimal propellant consumption. The considered flight sequence performs two boosts. The first one steers the launcher to a given GTO orbit. Then, after a ballistic flight, a second boost is used to perform the orbit transfer maneuver from the GTO to the GEO. We decompose the global optimization problem into two optimal control problems and we apply in this context the Hamilton-Jacobi-Bellman (HJB) approach together with a parameter optimization procedure.

Key words. optimal trajectory, optimal control problem, Hamilton-Jacobi approach, reachability analysis.

1 Introduction

This paper concerns the design of a global trajectory optimization procedure for a space shuttle of type Ariane 5, with the aim of steering a given payload from Earth to the GEO orbit. Trajectory optimization for aerospace launchers has been extensively studied in the literature, see for instance [3, 9, 12, 2] and the references therein.

Here it is assumed that the launcher performs two boosts. A first boost steers the launcher to a given GTO. Then, after a ballistic phase, a second boost is used to perform the transfer maneuver from the GTO to the GEO. The optimization problem aims at minimizing the final consumption of the launcher. More precisely, the complete flight sequence is composed of 4 important phases (atmospheric phase, propulsion with first stage until exhaustion of the ergol, propulsion with the first stage until injection on a GTO, ballistic flight until injection on GEO).

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Our approach is based on the reformulation of the trajectory optimization problem as a reachability problem that we solve by using an efficient method based on HJB approach in optimal control theory [10, 1]. A similar approach has been introduced by the authors in [5] and [4] where the orbital transfer maneuver from the GTO to the GEO is assumed to be of impulsive type. This hypothesis allows to estimate the amount of propellant needed to the orbital transfer using an explicit Tseolkovsky formula. In the present paper the problem is considered in more complete setting where a boost is needed to perform the orbital transfer. The duration of each boost is not fixed and is considered as an optimization parameter. The problem involves two more optimization parameters that are the shooting azimuth and the inclination speed. In addition to these parameters, the launcher is controlled by two functions; namely, the incidence and sideslip angles (time-dependent functions) that are also optimization variables.

This challenging problem presents several difficulties for the implementation of Hamilton-Jacobi-Bellman (HJB) approach. In the previous work [5] the authors have developed an interesting approach to combine the HJB framework with parameter optimization methods without increasing the state space dimension. it is shown here that this method can be applied for the general two-boost problem. Moreover, let us stress on that the proposed optimization procedure is supported by theoretical results that show the convergence of the procedure to a global optimal solution (see the references in [5]).

2 Physical model

Let O be the center of the Earth. Denote by r_T the Earth's mean radius. Consider the frame $\mathcal{R}_I = (O, \vec{i}_I, \vec{j}_I, \vec{k}_I)$ centered at O and such that the vector $\vec{k}_I$ is co-linear with the North-South axis of rotation, the vector $\vec{i}_I$ is located in the equatorial plane of the Earth and points to the Greenwich meridian at an elected date set here as $t = 0$. The vector $\vec{j}_I$ completes the orthonormal frame. This frame is considered as inertial. Consider also the frame $\mathcal{R}_R = (O, \vec{i}_R, \vec{j}_R, \vec{k}_R)$ that coincides with $\mathcal{R}_I$ at time $t = 0$ and that is rotating with the Earth around the axis $\vec{k}_I = \vec{k}_R$ with the angular velocity Ω. Let G be the mass center of the vehicle and $\vec{V}$ it's velocity. The state of the spacecraft can be characterized by the Cartesian coordinates $(\vec{X}, \vec{V})$ in the frame $\mathcal{R}_R$ with

$$\vec{X} = \big(x,\ y,\ y,\ \big)\,, \qquad \vec{V} = \big(V_x,\ V_y,\ V_z,\ \big)\,.$$

Let us define also the vertical local frame $\mathcal{R}_V = (G, \vec{i}_V, \vec{j}_V, \vec{k}_V)$ centered at G such that $\vec{k}_V$ is co-linear with $\vec{r}_G$ and pointing in the same direction. The vector $\vec{j}_V$ is in the orthogonal plane to $\vec{k}_V$ and pointing to the local North. The third vector $\vec{i}_V := \vec{j}_V \wedge \vec{k}_V$ is defined in such way to complete the orthonormal frame. The launcher may be also represented by the spherical coordinates of the position of G in $\mathcal{R}_R$ and those of the velocity in the frame $\mathcal{R}_V$.

These coordinates will be denoted by $(r, L, l, v, \chi, \gamma)$. The formulas that allow to pass from one coordinate system into another are classical and can be found in [11]. The total mass of the launcher will be denoted as $m \geq 0$.

2.1 Axis, angles, and forces

The launcher is subject to different forces that we describe here below.

Gravitational force: The gravitational field is defined as $\overrightarrow{g} = -\frac{\mu}{r^2}\left[I + J_2\left(\frac{r_T}{r}\right)^2 A\right] e_r$, where μ is the Earth's gravitational constant, J_2 is the second order term of the harmonic expansion of the gravitational field and I is identity matrix. The matrix A in the definition of the gravitational field is defined as follows:

$$A = \begin{pmatrix} 1 - 5\sin^2(l) & 0 & 0 \\ 0 & 1 - 5\sin^2(l) & 0 \\ 0 & 0 & 3 - 5\sin^2(l) \end{pmatrix}$$

So that the gravitational force is $\overrightarrow{F_g} = m\overrightarrow{g}$.

Aerodynamic forces: It is assumed here that the plane of symmetry of the vehicle coincides with the plane $(\vec{i}_V, \vec{k}_V)$ in the reference frame $\mathcal{R}_V$. The aerodynamic forces are:

- The drag force: $\overrightarrow{F_D}$ opposite to the velocity $\overrightarrow{V}$.
- The lift force: $\overrightarrow{F_L} \simeq 0$ is neglected in this application due to the technical specifications of the considered launcher.

Thrust force: It is assumed that the direction of the thrust force coincides with the axis of the launcher. The orientation of the thrust is defined by the incidence angle α and the sideslip angle δ. The modulus of the thrust force is given by $F_T(r) = \beta g_0 I_{sp} - SP(r)$ where $g_0 = 9.81$ ms^{-2}, $P(r)$ is the atmospheric pressure, β is the propellant flow rate, I_{sp} is the specific impulse and S is the surface.

Coriolis force $\overrightarrow{F_{CO}}$ and centripetal force $\overrightarrow{F_{CP}}$. These functions are defined by:

$$\overrightarrow{F_{CO}} = 2m\overrightarrow{\Omega} \wedge \overrightarrow{V} \quad \text{and} \quad \overrightarrow{F_{CP}} = m\overrightarrow{\Omega} \wedge (\overrightarrow{\Omega} \wedge \overrightarrow{OG}),$$

where $\overrightarrow{\Omega}$ is the Earth's angular velocity. These two forces are important to be taken into account as far as the launchers's trajectory is represented in a relative reference frame and not in the inertial one.

2.2 Motion's equations

It follows from the Newton's law of the motion that:

$$\frac{d\vec{X}}{dt} = \vec{V},\ m\frac{d\vec{V}}{dt} = \overrightarrow{F_g} + \overrightarrow{F_D} + \overrightarrow{F_L} + \overrightarrow{F_T} - \overrightarrow{F_{CO}} - \overrightarrow{F_{CP}}. \tag{1}$$

Let us denote

$$\begin{cases} F_v^C &= \Omega^2 r\cos\ell(\sin\gamma\cos\ell - \cos\gamma\sin\ell\cos\chi) \\ F_\gamma^C &= 2\Omega\cos\ell\sin\chi \\ & \quad + \frac{\Omega^2 r\cos\ell(\cos\gamma\cos\ell + \sin\gamma\sin\ell\cos\chi)}{v} \\ F_\chi^C &= \frac{\Omega^2 r}{v}\frac{\sin\ell\cos\ell\sin\chi}{\cos\gamma} - 2\Omega\,(\sin\ell - \tan\gamma\cos\ell\cos\chi) \end{cases}$$

the components of the vector $\overrightarrow{F^C} = -\overrightarrow{F_{CO}} - \overrightarrow{F_{CP}}$ in spherical coordinates. Straightforward calculations yield to the motion's equation in the spherical coordinates:

$$\begin{aligned}
\frac{dr}{dt} &= v\sin\gamma \\
\frac{d\ell}{dt} &= \frac{v}{r}\cos\gamma\cos\chi \\
\frac{dv}{dt} &= -g_r\sin\gamma + g_\ell\cos\gamma\cos\chi + \frac{F_T(r)}{m}\cos\alpha\cos\delta \\
& -\frac{F_D(r,v,\alpha)}{m} + F_v^C \\
\frac{d\gamma}{dt} &= -\cos\gamma\left(\frac{g_r}{v} - \frac{v}{r}\right) - \sin\gamma\cos\chi\frac{g_\ell}{v} - \frac{F_T(r)}{mv}\sin\alpha \\
& + F_\gamma^C \\
\frac{d\chi}{dt} &= \frac{-g_\ell\sin\chi}{v\cos\gamma} - \frac{v}{r}\cos\gamma\tan\ell\sin\chi + \frac{F_T(r)\cos\alpha\sin\delta}{m\,v\cos(\gamma)} \\
& + F_\chi^C
\end{aligned} \tag{2}$$

where we have isolated the differential equation for the component L:

$$\frac{dL}{dt} = \frac{v}{r}\frac{\cos\gamma\sin\chi}{\cos\ell}. \tag{3}$$

2.3 Mass evolution

The evolution of the mass $m(t)$ is also needed in order to get a complete model. This evolution is given by the ODE:

$$\dot{m}(t) = \beta(t) \tag{4}$$

where the function β is known and represents the consumption flow rate and depends on the launcher's parameters. In the sequel, we denote by $\beta_{EAP} > 0$, $\beta_{E_1} > 0$ and $\beta_{E_2} > 0$ the mass flow rates for the boosters, the first and the second stage of the launcher respectively. For sake of simplicity, we shall consider that these rates are known constants.

2.4 Dynamical system

As mentioned in the previous subsection, the motion of the Longitude variable L can be isolated from the other variables (indeed, none of the other variables depend on L). So, in the sequel, the state variable will be defined as $(x,m) \in \mathbb{R}^6,$ with $x = (r,l,v,\chi,\gamma)$. During the phases where one or more engines are on, the system is controlled by the direction of the thrust force. Let

$$U := [\alpha_{min}, \alpha_{\max}] \times [\delta_{min}, \delta_{max}],$$

be the control set with constants $\alpha_{min/max}$ and $\delta_{min/max}$ such that $[\alpha_{min}, \alpha_{\max}] \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[\delta_{min}, \delta_{\max}] \subset [-\pi, \pi]$. The control variable vector will be denoted by $u = (\alpha, \delta) \in U$. Set $\mathcal{U}_{ad}$ as the set of all admissible controls:

$$\begin{aligned} \mathcal{U}_{ad} \quad := \quad & \{\mathbf{u} := (\alpha(.), \delta(.)) : \text{ measurable function} \\ & \text{on } [0, +\infty) \text{ that takes values in } U\}. \end{aligned}$$

The differential equations governing the evolution of the dynamical system depend on the phases of the flight.

Phase 0. The mission of the launcher starts when the vehicle leaves the launch base. Both boosters along with the stage E_1 are ignited and consume propellant with flow rates β_{EAP} and β_{E_1} respectively. The initial mass of the launcher is known:

$$\begin{aligned} M_{init} \quad := \quad & \underbrace{M_{\mathrm{EAP}} + M_{\mathrm{E1}} + M_{\mathrm{E2}} + m_{\mathrm{F}}}_{Structure} \\ & + \underbrace{m_{\mathrm{CU}}}_{Payload} + \underbrace{M_{\mathrm{P,EAP}} + M_{\mathrm{P,E1}} + M_{\mathrm{P,E2}}}_{Propellant}. \end{aligned} \tag{5}$$

The trajectory of the spacecraft during this phase is defined by two parameters : ψ_0, the shooting azimuth and $\omega_{\mathtt{basc}}$, constant inclination speed. In this phase of the flight, no further control is acting on the launcher, see [5] for more details. The values of the parameters ψ_0 and $\omega_{\mathtt{basc}}$ are unknowns that must be determined in such a way to optimize the launcher's consumption. The boosters are ejected at a given time t_0, once all the propellant is consumed. The set of possible positions corresponding to a large sample of the parameters $(\psi_0, \omega_{\mathtt{basc}})$ can be obtained by a simple integration of the motion's equations. Let us denote by X_0 the set of all possible states of the launcher at time t_0 (the end of phase 0) for all values of parameters $(\psi_0, \omega_{\mathtt{basc}})$ in a given set $P_{\mathtt{Ini}}$. Each point of X_0 can be considered as image by a known application Ψ of shooting parameters (ψ, ω):

$$x \in X_0 \;\Leftrightarrow\; \exists p = (\psi, \omega) \in P_{\mathtt{Ini}}, x = \Psi(p).$$

The total mass of the launcher at time t_0 (the end of phase 0) can be explicitly calculated:

$$\begin{aligned} m(t_0) \quad = \quad & M_0 = \underbrace{M_{\mathrm{E1}} + M_{\mathrm{E2}} + m_{\mathrm{F}}}_{Structure} \\ & + \underbrace{m_{\mathrm{CU}}}_{Payload} + \underbrace{M_{\mathrm{P,E1}} - \beta_{E_1} t_0 + M_{\mathrm{P,E2}}}_{Propellant}. \end{aligned} \tag{6}$$

Remark, that at the end of this phase the launcher is assumed to be out of the atmosphere. So, for all the following phases the aerodynamic forces can be neglected.

Boost 1. This part of the mission starts from the set X_0 and ends when the launcher has reached the given injection region on the transfer orbit, GTO. Denote $\mathcal{C}$ the set of possible injection states. The following hypothesis will be assumed

Hypothesis 1 (Properties of the set $\mathcal{C}$) *The set $\mathcal{C}$ is a segment of a single GTO orbit around of its perigee.*

During the first boost, the engines of the first stage are on until the complete propellant consumption of the first stage. The first stage is ejected at a known time t_1. Then the engines of the second stage are on, during the time interval $[t_1, t_2]$. The evolution of the launcher is described by the following system of differential equations

$$F_1 : \begin{cases} \frac{d\mathbf{x}}{dt} = f_1(t, \mathbf{x}(t), \mathbf{m}(t), \mathbf{u}_1(t)),\ t \in]t_0, t_2[\\ \frac{d\mathbf{m}}{dt} = \beta_1(t) \\ \mathbf{x}(t_0) \in X_0,\ \mathbf{m}(t_0) = M_0,\ \mathbf{u}_1 \in \mathcal{U}_{ad} \end{cases} \tag{7}$$

where $f_1(t, x, m, u)$ is the function corresponding to the right hand side of the system of differential equations (2). The thrust force and propellant flux functions are defined as follows:

$$F_T(t) = \begin{cases} \beta_{E_1} g_0 I_{sp}, t \in [t_0, t_1] \\ \beta_{E_2} g_0 I_{sp}, t \in [t_1, t_2] \end{cases}, \beta_1(t) = \begin{cases} \beta_{E_1}, t \in [t_0, t_1] \\ \beta_{E_2}, t \in [t_1, t_2]. \end{cases}$$

The end time t_2 of the first boost is unknown and may be optimized. It can be bounded by the known constant $T_2 = t_1 + \dfrac{M_{\mathrm{P,E2}}}{\beta_{E_2}}$ corresponding to the maximal duration of the propellant consumption for the engine of the second stage. So one has: $t_0 < t_1 < t_2 \leq T_2$. Let us define

$$M_1 = \underbrace{M_{\mathrm{E2}}}_{Structure} + \underbrace{m_{\mathrm{CU}}}_{Payload} + \underbrace{M_{\mathrm{P,E2}}}_{Propellant} \tag{8}$$

the total mass of the launcher at the time t_1, after ejection of the first stage. At the end time t_2 of the first boost, the following state constraint should be satisfied :

$$x(t_2) \in \mathcal{C},\ m(t_2) \geq 0 \tag{9}$$

where $\mathcal{C}$ is the injection coordinates subset of a given GTO orbit (satisfying (H1)).

The ballistic flight. Once the launcher has reached a GTO, the engine of the second floor is off. Then follows a ballistic flight phase. The duration of this phase is not fixed and depends on the GTO parameters. Denote $t_3 > t_2$ the end time of this phase. The trust force is zero because all engines are off. Therefore, the launcher's motion is governed by an uncontrolled and autonomous differential system:

$$G : \begin{cases} \frac{d\mathbf{x}}{dt} = g(\mathbf{x}(t), \mathbf{m}(t))\ t \in]t_2, t_3] \\ \frac{d\mathbf{m}}{dt} = 0 \end{cases} \tag{10}$$

where $g(x, m)$ is the function corresponding to the right hand side of the system of differential equations (2) with zero thrust force. Let us denote $\Phi_g(t, (x, m))$ the associated flow map, such that for each (x, m) the function $\Phi_g(\cdot, (x, m)$ is solution of (10) with initial condition $\Phi_g(t_2, (x, m)) = (x, m)$.

Boost 2. This phase can start when the launcher has reached a position near the apogee of the GTO. The second engine is ignited to perform the orbital transfer to GEO. Once the GEO reached at some unknown time $t_f > t_3$, the mission is completed. During this phase, the state equations are controlled by the direction of the thrust force and are autonomous:

$$F_2 : \begin{cases} \frac{d\mathbf{x}}{dt} = f_2(\mathbf{x}(t), \mathbf{m}(t), \mathbf{u}_2(t)), \ t \in]t_3, t_f[\\ \frac{d\mathbf{m}}{dt} = \beta_{E_2}. \\ \mathbf{x}(t_f) \in \mathcal{C}_f, \ \mathbf{u}_2 \in \mathcal{U}_{ad} \end{cases} \tag{11}$$

where $f_2(x, m, u)$ is the function corresponding to the right hand side of the system of differential equations (2) with constant thrust force: $F_T = \beta_{E_2} g_0 I_{sp}$.

3 Optimal control problem

Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{U}_{ad} \times \mathcal{U}_{ad}$ and $x \in X_0$. An admissible trajectory corresponding to $\mathbf{u}$ and x is the absolutely continuous function $(\mathbf{x}(\cdot), \mathbf{m}(\cdot)) = \left(\mathbf{x}^{\mathbf{u},t_2,t_3}_{(x,M_0)}(\cdot), \mathbf{m}^{\mathbf{u},t_2,t_3}_{(x,M_0)}(\cdot)\right)$ defined on $[t_0, t_f[$ such that

- $\exists t_1 < t_2 < T_2$ such that $(\mathbf{x}(\cdot), \mathbf{m}(\cdot))$ is solution of (7) associated to the control $\mathbf{u}_1(\cdot)$ on the time interval $[t_0, t_2[$ with initial condition $\mathbf{x}(t_0) = x, \ \mathbf{m}(t_0) = M_0$;
- At time t_2, the condition (9) is satisfied;
- $\exists t_3 > t_2$ such that $(\mathbf{x}(\cdot), \mathbf{m}(\cdot))$ is solution of (10) on the time interval $[t_2, t_3[$
- $\exists t_f > t_3$ such that $(\mathbf{x}(\cdot), \mathbf{m}(\cdot))$ is solution of (11) on the time interval $[t_3, t_f[$ associated to the control $\mathbf{u}_2(\cdot)$ and satisfying the terminal condition $\mathbf{x}(t_f) \in \mathcal{C}_f$.

The optimal control problem aims to maximize the final propellant mass at the end of the mission.

$$\begin{cases} \text{Maximize } \mathbf{m}^{\mathbf{u},t_2,t_3}_{(x,M_0)}(t_f) \\ \quad \left(\mathbf{x}^{\mathbf{u},t_2,t_3}_{(x,M_0)}, \mathbf{m}^{\mathbf{u},t_2,t_3}_{(x,M_0)}\right) \text{ is an admissible solution} \\ \qquad\qquad\qquad \text{associated with } (\mathbf{u}, x, t_2, t_3) \\ \quad \mathbf{u} \in \mathcal{U}_{ad} \times \mathcal{U}_{ad}, \ t_1 < t_2 < t_3 < t_f, \\ \quad x \in X_0, \ \mathbf{x}^{\mathbf{u},t_2,t_3}_{(x,M_0)}(t_2) \in \mathcal{C}, \ \text{and } \mathbf{x}^{\mathbf{u},t_2,t_3}_{(x,M_0)}(t_f) \in \mathcal{C}_f. \end{cases} \tag{$\mathcal{P}$}$$

4 HJB approach

First, the optimal control problem $(\mathcal{P})$ is characterized with two reachability problems and a parameter optimization. As explained in the section (2.4) the optimization over the set of shooting parameters is equivalent to the optimization over the set of initial states X_0. The other parameters of the problem are the durations $\tau_2 = t_2 - t_1$ and $\tau_3 = t_3 - t_2$ of the first boost and of the ballistic flight respectively. It is assumed here that these parameters take their values in some known intervals: $\tau_2 \in I_{\tau_2} = \left[0, \dfrac{M_{\mathrm{P,E2}}}{\beta_{E_2}}\right]$ and $\tau_3 \in I_{\tau_3} = \left[\tau_3^{min}, \tau_3^{min}\right]$.

4.1 Reachability problems

Let us associate with two controlled phases of the flight the following reachability problems.

Definition 4.1 *Let $(\mathbf{x}^{\mathbf{u}_2}_{(x,m)}, \mathbf{m}^{\mathbf{u}_2}_{(x,m)})$ be a solution of (11) starting at (x, m) at time $\tau = 0$ and associated with an admissible control $\mathbf{u}_2 \in \mathcal{U}_{ad}$. The backward reachable set of duration τ for the system (11) and the target $\mathcal{C}_f$ is the set:*

$$\begin{aligned}\mathcal{R}^{\mathcal{C}_f}_{F_2}(\tau) \quad = \quad & \Big\{(x,m) \;\Big|\; \exists \mathbf{u}_2 \in \mathcal{U}_{ad} \textit{ s.t. } \mathbf{x}^{\mathbf{u}_2}_{(x,m)}(\tau) \in \mathcal{C}_f, \\ & \textit{and } \mathbf{m}^{\mathbf{u}_2}_{(x,m)}(\tau) \geq 0\Big\}. \end{aligned} \tag{12}$$

Set $\mathcal{D} := \Phi(I_{\tau_3}, \mathcal{C} \times [0, M_0])$. By definition of the dynamical system (10), the hypothesis (H1) implies that the set $\mathcal{D}$ is a segment of the same orbit. So, one can notice that

$$\forall x \in \mathcal{C},\ \forall (y,m) \in \mathcal{D},\ \exists \tau_3 \in I_{\tau_3} \text{ s.t. } (y,m) = \Phi(\tau_3, (x,m)).$$

On another hand, since the dynamics F_2 is autonomous, the backward reachable set $\mathcal{R}^{\mathcal{C}_f}_{F_2}(\tau)$ does not depend on the initial time but only on the duration. Then one can define also the minimal duration function to reach the target.

Definition 4.2 *The minimal duration function associated with the backward reachability problem 4.1 is the application:*

$$\mathcal{T}_{F_2}(x,m) = \inf\Big\{\tau \in [0, \tau_f^{max}] \;\Big|\; (x,m) \in \mathcal{R}^{\mathcal{C}_f}_{F_2}(\tau)\Big\}. \tag{13}$$

Definition 4.3 *Let $(\mathbf{x}^{\mathbf{u}_1}_{(x,m)}, \mathbf{m}^{\mathbf{u}_1}_{(x,m)})$ be a solution of (7) starting at (x, m) at time $t = t_0$ and associated with an admissible control $\mathbf{u}_1 \in \mathcal{U}_{ad}$. The backward reachable set at time t_2 for the system F_1 defined by (7) and the target $\mathcal{C}$ the set:*

$$\begin{aligned}\mathcal{R}^{\mathcal{C}}_{F_1}(t_2) \quad = \quad & \Big\{(x,m) \;\Big|\; \exists \mathbf{u}_1 \in \mathcal{U}_{ad} \textit{ s. t. } \mathbf{x}^{\mathbf{u}_1}_{(x,m)}(t_2) \in \mathcal{C}, \\ & \textit{and } \mathbf{m}^{\mathbf{u}_1}_{(x,m)}(t_2) \geq 0\Big\} \end{aligned} \tag{14}$$

From the above definitions it is clear that the following result holds.

Theorem 4.1 *Consider an initial position $x \in X_0$ and fix $(\tau_2, \tau_3) \in I_{\tau_2} \times I_{\tau_3}$. Set $t_2 = t_1 + \tau_2$ and $t_3 = t_2 + \tau_3$. Then for a given control $\mathbf{u} \in \mathcal{U}_{ad}$ there exists an admissible trajectory $(\mathbf{x}, \mathbf{m}) = \left(\mathbf{x}^{\mathbf{u},t_2,t_3}_{(x,M_0)}, \mathbf{m}^{\mathbf{u},t_2,t_3}_{(x,M_0)}\right)$ if and only if*

$$\begin{aligned} &(i) \qquad (x, M_0) \in \mathcal{R}^{\mathcal{C}}_{F_1}(t_2) \\ &(ii) \qquad \exists t_f \;\Big|\; \Phi_g\left(\tau_3, (\mathbf{x}(t_2), \mathbf{m}(t_2))\right) \in \mathcal{R}^{\mathcal{C}_f}_{F_2}(t_f - t_3). \end{aligned} \tag{15}$$

Then, for any admissible trajectory one can write explicitly the corresponding final mass $\mathbf{m}^{\mathbf{u},t_2,t_3}_{(x,M_0)}(t_f)$:

$$\begin{aligned}\mathbf{m}^{\mathbf{u},t_2,t_3}_{(x,M_0)}(t_f) \quad &= \quad \mathbf{m}^{\mathbf{u},t_2,t_3}_{(x,M_0)}(t_2) - \beta_{E_2}(t_f - t_3) \\ &= \quad M_1 - \beta_{E_2}(t_2 - t_1) - \beta_{E_2}(t_f - t_3) \end{aligned} \tag{16}$$

Due to the hypothesis (1) the terminal time t_f in (15) does not depend on $\mathbf{x}(t_2)$ but only on $\mathbf{m}^{\mathbf{u},t_2,t_3}_{(x,M_0)}(t_2)$.

4.2 Solution algorithm

To solve the problem $(\mathcal{P})$ the sets of parameters are discretized taking uniform grids. Let $P^{\Delta}_{\mathtt{Ini}} = \{(\psi_i, \omega_j),\ i,j = 1,\dots,N_I\}$ be a subset of $N_I \times N_I$ values in P_{Ini} and X_0^{Δ} the corresponding finite subset of initial positions (see [5] for more details about the computation procedure). Let $\Xi^{\Delta} = \{(\tau_2^i, \tau_3^j),\ i,j = 1,\dots,N_\tau\}$ be a discrete subset of $I_{\tau_2} \times I_{\tau_3}$. It is assumed that, when the discretization mesh step Δ tends to 0, the sets X_0^{Δ} and Ξ^{Δ} tend respectively to X_0 and $I_{\tau_2} \times I_{\tau_3}$, in the sense of convergence of sets. Let $\mathcal{C}^{\Delta}$ be the discrete approximation of the set $\mathcal{C}$ of injection states on the GTO and $\mathcal{D}^{\Delta}$, the set of images of $\mathcal{C}^{\Delta}$ by the flow map Φ_g. The algorithm for solving the optimal control problem $(\mathcal{P})$ is the following:

Step 1- Solve the minimum time optimal control problem for the system F_2 and compute the minimum time function (13).
Step 2- For any $(\tau_2, \tau_3) \in \Xi^{\Delta}$ solve the reachability problem (14) with $t_2 = t_1 + \tau_2$.
Step 3-For every $x \in X_0^{\Delta}$ such that $(x, M_0) \in \mathcal{R}^{\mathcal{C}}_{F_1}(t_2)$ compute the corresponding trajectory $\left(\mathbf{x}^{\mathbf{u}_1}_{(x,M_0)}, \mathbf{m}^{\mathbf{u}_1}_{(x,M_0)}\right)$ solution of (7) such that $\left(\mathbf{x}^{\mathbf{u}_1}_{(x,M_0)}, \mathbf{m}^{\mathbf{u}_1}_{(x,M_0)}\right)(t_2) \in \mathcal{C}$ and compute, by direct minimization

$$M_2(x, t_2) = \beta_{E_2} \min_{(y,\mathbf{m}(t_2)) \in \mathcal{D}^{\Delta}} \beta_{E_2} \mathcal{T}_{F_2}(y, \mathbf{m}(t_2))$$

Step 4 Compute by direct maximization over the set of parameters (x, τ_2) the optimal final mass

$$\sup_{x \in X_0, \tau_2 \in I_{\tau_2}} \left[M_1 - \beta_{E_2}(t_2 - t_1) - M_2(x, t_2)\right]$$

4.3 HJB approach to solve reachability problems

4.3.1 Characterization of the reachable set $\mathcal{R}^{\mathcal{C}}_{F_1}(t_2)$.

This problem can be solved with the approach developed in [6] for the reachability problems with non-autonomous dynamical systems. Indeed, when the time horizon t_2 is known, one can easily rewrite the system F_1 under the following form

$$\widetilde{F_1} : \begin{cases} \frac{d\mathbf{x}}{dt} = \widetilde{f_1}(t_2 - t, \mathbf{x}(t), \mathbf{m}(t), \mathbf{u}_1(t)),\ t \in]t_0, t_2[\\ \frac{d\mathbf{m}}{dt} = \widetilde{\beta_1}(t_2 - t) \\ \mathbf{x}(t_0) \in X_0,\ \mathbf{m}(t_0) = M_0,\ \mathbf{u}_1 \in \mathcal{U}_{ad}. \end{cases}$$

Let φ be a Lipschitz continuous function that characterize the target set $\mathcal{C}$ as follows: $x \in \mathcal{C} \Leftrightarrow \varphi(x) \le 0$. Typically, one can use the signed distance function. Then, following [6], one can define the value function

$$v_1(x, m, t_2) = \inf_{\mathbf{u}_1} \varphi(\mathbf{x}^{\mathbf{u}_1}_{(x,m)}(t_2)) \vee (-\mathbf{m}^{\mathbf{u}_1}_{(x,m)}(t_2)) \tag{17}$$

where $a \vee b = \max\{a, b\}$. It is shown in [6] that this value function is a unique Lipschitz continuous solution of

$$\frac{\partial v_1}{\partial t} + H_1(t, x, m, D_{(x,m)} v_1(x, m, t)) = 0$$
$$v_1(x, m, 0) = \varphi(x) \vee (-m) \tag{18}$$

with $H_1(t, x, m, p) = \max_{u_1 \in U} -p \cdot \widetilde{F_1}(t, x, m, u)$ and that this function characterize the reachable set $\mathcal{R}^{\mathcal{C}}_{F_1}(t_2)$ as follows: $(x, m) \in \mathcal{R}_{F_1}(\mathcal{C}, t_2) \Leftrightarrow v_1(x, m, t_2) \leq 0$.

4.3.2 Characterization of the minimal time function $\mathcal{T}_{F_2}(x, m)$.

here again the approach introduced in [8] will be used. The system F_2 is autonomous. Let ψ be a Lipschitz continuous function that characterize the target set $\mathcal{C}_f$ as follows: $x \in \mathcal{C}_f \Leftrightarrow \psi(x) \leq 0$. Then, following [8], one can define the value function

$$v_2(x, m, \tau) = \inf_{\mathbf{u}_2} \psi(\mathbf{x}^{\mathbf{u}_2}_{(x,m)}(\tau)) \vee (-\mathbf{m}^{\mathbf{u}_2}_{(x,m)}(\tau)). \tag{19}$$

According to [8], $v_2(x, m, \tau)$ is a unique Lipschitz continuous solution of

$$\frac{\partial v_2}{\partial \tau} + H_2(x, m, D_{(x,m)} v_2(x, m, \tau)) = 0$$
$$v_2(x, m, 0) = \psi(x) \vee (-m) \tag{20}$$

with $H_2(x, m, p) = \max_{u \in U} -p \cdot F_2(x, m, u)$. This function characterizes the minimum time function as follows : $\mathcal{T}_{F_2}(x, m) = \min\{\tau > 0 \mid v_2(x, m, \tau) \leq 0\}$.

5 Numerical results

This section presents some numerical results obtained using ROCHJ (see [7]) software for solving HJB equations. The data used for the simulations was provided by CNES together with an example of optimal trajectory calculated by CNES using Shooting methods. The considered launcher is an Ariane-type two-stage spacecraft. The numerical data used in all the computations are the same as in [5].

After the computation of the value functions v_1 and v_2 (by solving a HJB equation for each one), an optimal trajectory for a given initial condition (x, M_0) is computed by following the proposed solution algorithm as follows:

Step 1. Compute a trajectory $(\mathbf{x}^{\mathbf{u}_1}_{(x,M_0)}, \mathbf{m}^{\mathbf{u}_1}_{(x,M_0)})$ that reaches the target GTO set $\mathcal{C}$ at time t_2, using the value function v_1. (See [5] for more details about reconstruction algorithm).
Step 2. Integrate the motion equations for the ballistic flight (10) starting from $(\mathbf{x}^{\mathbf{u}_1}_{(x,M_0)}(t_2), \mathbf{m}^{\mathbf{u}_1}_{(x,M_0)}(t_2))$ until a given time $t_3 > t_2$.

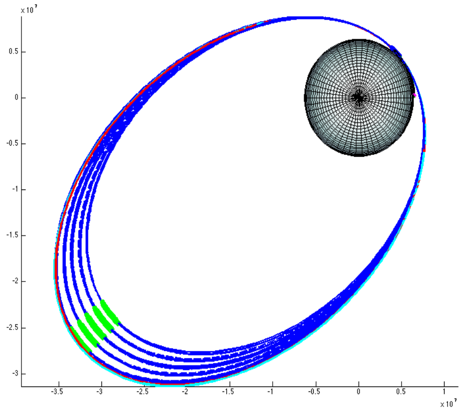

Figure 1: Balistic flight prolongations of the first boost trajectories for different grids

Step 3. Reconstruct, using the minimum time function (13), the time-optimal trajectory $(\mathbf{x}^{\mathbf{u}_2}_{(x,\mathbf{m}(t_2))}, \mathbf{m}^{\mathbf{u}_2}_{(x,\mathbf{m}(t_2))})$.

Note first that the approach presented in this paper is a global approach that provides the optimal trajectory (with the best performance). This is an advantage compared to other numerical approaches based on classical optimization algorithms which provide generally only local solutions. The approach doesn't require any particular initialization of the computation, unlike the approaches based on shooting algorithms which require a good initial guess to guarantee the convergence of the computational process.

The quality of our approach depends on the computation grids used to solve the HJB equations. In Fig. 2, one can see that the approximated trajectories are quite similar during Phase 0 and Boost 1. They are all close to the reference solution, and the error approximations decrease as the size of the grids increase. The following grids were tested: grid 1: $20 \times 8 \times 20 \times 20 \times 16 \times 5$, grid 2: $30 \times 12 \times 30 \times 30 \times 24 \times 5$, grid 3: $40 \times 16 \times 40 \times 40 \times 32 \times 5$, and grid 4: $50 \times 20 \times 50 \times 50 \times 40 \times 5$. The Fig. 1 shows the trajectories obtained by integration of the ballistic flight equations (10) from the end points of the first boost trajectories obtained from different computational grids. One can notice that only the more finest computational grid allows to obtain a good approximation of the target GTO such that the ballistic flight phase reaches a segment of GEO and allows to perform a good transfer maneuver.

It should be pointed out that the cost of computation depends also on the grid used for solving the HJB equations. The presented results were obtained using a computer with Intel XEON E5-2695 (24 Cores) processor and a multi-threaded ROC-HJ Sofware. The whole numerical simulations on a grid 1 take about 450 seconds, while on a more finest grid 4, the computation takes 6 hours.

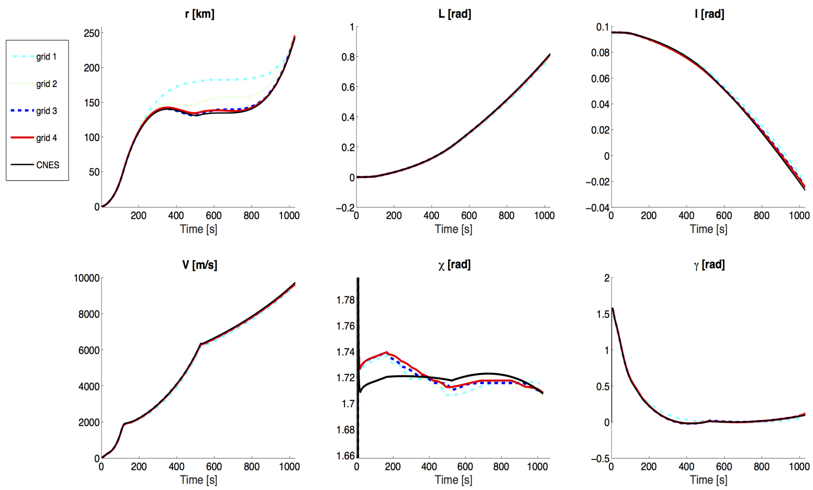

Figure 2: Optimal trajectories with different grids in spherical coordinates: atmospheric flight and boost 1

To compute the global optimal trajectory one needs to solve the corresponding HJB equations (18) and (20) on a sufficiently fine grids. The Fig. 3 shows the obtained optimal two-boost trajectory, starting on the Earth and arriving on the GEO with a small propellant mass gain: $m(t_f) = 200kg$.

6 Conclusion

The paper introduces a new mathematical formulation of the launching problem that leads to a global optimization procedure based on Hamilton-Jacobi-Bellman approach. Recent theoretical and numerical development in HJB approach allow to tackle this trajectory optimization problem including complex flight phases with two boosts and a ballistic flight. Other optimization procedures have already been extensively studied and used in aerospace literature, and their features are very well known (precision, convergence, ...). However these approaches provide (in general) local optima and their convergence are very sensitive to the initialization of the optimization algorithms. The Hamilton-Jacobi approach turns out to be quite effective and provides a global solution. Even though, it should be noticed that the HJB approach requires a CPU time that increases as the size of the grid increase.

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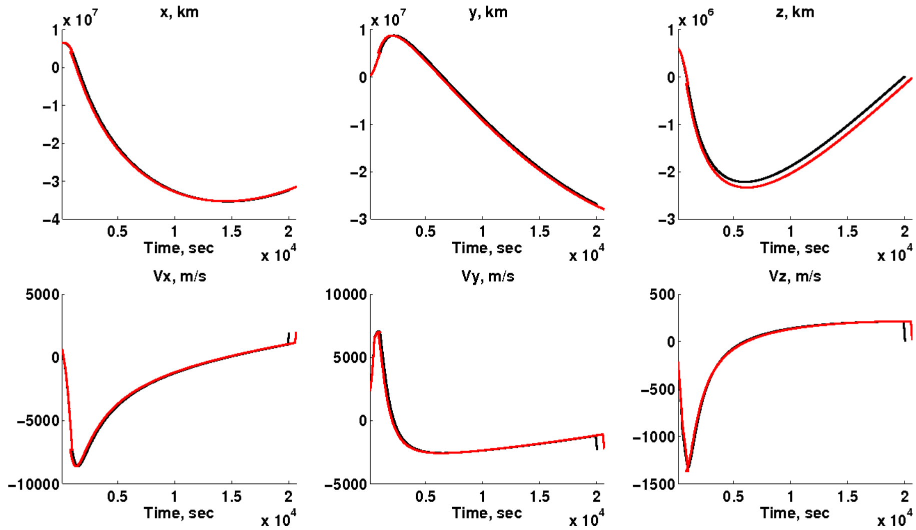

Figure 3: Optimal two-boost trajectory obtained with grid 4 (red) compared to the reference CNES trajectory in inertial cartesian reference frame

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