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February 10, 2021

Abstract

In this paper, we study input-to-state stability (ISS) issues for damped wave equations with Dirichlet boundary conditions on a bounded domain of dimension two. The damping term is assumed to be non-linear and localized to an open subset of the domain. We handle the disturbed case as an extension of [16], where stability results are given with a damping term active on the full domain with no disturbances considered. We provide input-to-state types of results.

1 Introduction

Consider the damped wave equation with localized damping, with Dirichlet boundary conditions given by

$$(\mathbf{P_{dis}}) \quad \begin{cases} u_{tt} - \Delta u = -a(x)g(u_t + d) - e, & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u(0,.) = u^0 \ , \ u_t(0,.) = u^1, & \end{cases} \tag{1}$$

where Ω is a C^2 bounded domain of $\mathbb{R}^2$, d and e stand for a damping disturbance and a globally distributed disturbance for the wave dynamics respectively. The term $-a(x)g(u_t+d)$ stands for the (perturbed) damping term, where $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1 non-decreasing function verifying $\xi g(\xi) > 0$ for $\xi \neq 0$ while $a : \overline{\Omega} \to \mathbb{R}$ is a continuous non negative function which is bounded below by a positive constant a_0 on some non-empty open subset ω of Ω. Here, ω is the region of the domain where the damping term is active, more precisely, the region where the localization function a is bounded from below by a_0. As for the initial condition (u^0, u^1), it belongs to the standard Hilbert space $H_0^1(\Omega) \times L^2(\Omega)$.

In this paper, our aim is to obtain input-to-state stability (ISS) type of results for $(\mathbf{P_{dis}})$, i.e., estimates of the norm of the state u which, at once, show that trajectories tend to zero in the absence of disturbances and remain bounded by a function of the norms of the disturbances otherwise.

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*This research was partially supported by the iCODE Institute, research project of the IDEX Paris-Saclay, and by the Hadamard Mathematics LabEx (LMH) through the grant number ANR-11-LABX-0056-LMH in the "Programme des Investissements d'Avenir".

One can refer to [20] for a thorough review of ISS results and techniques for finite dimension systems and to the recent survey [19] for infinite dimensional dynamical systems. In the case of the undisturbed dynamics, i.e., (1) with $(d, e) \equiv (0, 0)$, there is a vast literature regarding the stability of the corresponding system with respect to the origin, which is the unique equilibrium state of the problem. This in turn amounts to have appropriate assumptions on a and g, cf. [3] for extensive references. We will however point out the main ones that we need in order to provide the context of our work. To do so, we start by defining the energy of the system by

$$E(t) = \frac{1}{2} \int_{\Omega} \left(|\nabla u|^2(t, \cdot) + u_t^2(t, \cdot) \right) \, dx, \tag{2}$$

which defines a natural norm on the space $H_0^1(\Omega) \times L^2(\Omega)$. Strong stabilization has been established in the early works [8] and [10], i.e., it is proved with an argument based on the Lasalle invariance principle that $\lim_{t \to +\infty} E(t) = 0$ for every initial condition in $H_0^1(\Omega) \times L^2(\Omega)$. However, no decay rate of convergence for E is established since it requires in particular extra assumptions on g and ω.

As a first working hypothesis, we will assume that $g'(0) > 0$, classifying the present work in those that aimed at establishing results of exponential convergence for strong solutions. We refer to [3] for the line of work where g is assumed to be super-linear in a neighborhood of the origin (typically of polynomial type). Note that, in most of these works (except for the linear case) the rate of exponential decay of E depends on the initial conditions. That latter fact in turn relies on growth conditions of g at infinity. Regarding the assumptions on ω, they have been first put forward in the pioneering work [21] on semi-linear wave equations and its extension in [13], where the multiplier geometric conditions (MGC) have been characterized for ω in order to achieve exponential stability. For linear equations, the sharpest geometrical results are obtained by microlocal techniques using the method of geometrical optics, cf [4] and [7].

In this paper, our objective is to obtain results for non-linear damping terms and one should think of the nonlinearity g not only as a mean to provide more general asymptotic behavior at infinity than a linear one but also as modeling an uncertainty of the shape of the damping term. Dealing with nonlinearities justifies why microlocal techniques are not suited here and we will be using the multiplier method as presented e.g. in [12]. Many results have been established in the case where $g'(0) = 0$, for instance, decay rates for the energy are provided in [15] in the localized case but the non-linearity is to have a linear growth for large values of its arguments. Note that the estimates as presented in [15] are not optimal in general, as for instance in the case of a power-like growth. For general optimal energy decay estimates and for general abstract PDEs, we refer the reader to [1] for a general formula for explicit energy decay estimates and to [2] for an equivalent simplified energy decay estimate with optimality results in the finite dimensional case. However, when it comes to working under the hypothesis $g'(0) > 0$, few general results are available. One can find a rather complete presentation of the available results in [16]. In particular, the proof of exponential stability along strong solutions has only been given for general nonlinearities g, in dimension two and in the special case of a non-localized damping with no disturbances requiring only one multiplier coupled with a judicious use of Gagliardo-Nirenberg's inequality. Our results generalize this finding in the absence of disturbances (even though it has been mentioned in [16] with no proof that this is the case). It has also to be noted that similar results are provided in [15] in the localized case but the nonlinearity is lower bounded by a linear function for large values of its arguments. That simplifies considerably some computations. Recall also that the purpose of [15] is instead to address issues when $g'(0) = 0$ and to obtain accurate decay rates for E.

Hence a possible interest of the present paper is the fact that it handles nonlinearities g so that $g(v)/v$ tends to zero as $|v|$ tends to infinity with a linear behavior in a neighborhood of the origin.

As for ISS purposes, this paper can be seen as an extension to the infinite dimensional context of [14] where the nonlinearity is of the saturation type. Moreover, the present work extends to the dimension two the works [17] and [18], where this type of issues have been addressed by building appropriate Lyapunov functions and by providing results in dimension one. Here, we are not able to construct Lyapunov functions and we rely instead on energy estimates based on the multiplier method, showing how these estimates change when adding the two disturbances d and e. To develop that strategy, we must impose additional assumptions on g', still handling saturation functions. As a final remark, we must recall that [16] contains other stability results in two directions. On one hand, g' can simply admit a (possibly) negative lower bound and on the other hand, the space dimension N can be larger than 2, at the price of more restrictive assumptions on g, in particular, by assuming quasi-linear lower bounds for its asymptotic behavior at infinity. One can readily extend the results of the present paper in both directions by eventually adding growth conditions on g.

2 Statement of the problem and main result

In this section, we provide assumptions on the data needed to precisely define (1). We henceforth refer to (1) as the disturbed problem ($\mathbf{P_{dis}}$). Next, we state and comment the main results of this work and discuss possible extensions.

Throughout the paper, the domain Ω is a bounded open subset of $\mathbb{R}^2$ of class C^2, the assumptions on g are the following.

($\mathbf{H_1}$): The function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1 non-decreasing function such that

$$g(0) = 0, \quad g'(0) > 0, \; g(x)x > 0 \;\text{ for } x \neq 0, \tag{3}$$

$$\exists\, C > 0, \; \exists\, 1 < q < 5, \; \forall\, |x| \geq 1, \; |g(x)| \leq C|x|^q, \tag{4}$$

$$\exists\, C > 0, \; \exists\, 0 < m < 4, \; \forall |x| > 1, \quad |g'(x)| \leq C|x|^m. \tag{5}$$

($\mathbf{H_2}$): The localization function $a : \overline{\Omega} \to \mathbb{R}$ is a continuous function such that

$$a \geq 0 \;\; on \;\; \Omega \quad \text{and} \;\; \exists\, a_0 > 0, \; a \geq a_0 \;\; on \;\; \omega. \tag{6}$$

In order to prove the stability of solutions, we impose a multiplier geometrical condition (MGC) on ω. It is given by the following hypothesis.

($\mathbf{H_3}$): There exists an observation point $x_0 \in \mathbb{R}^2$ for which ω contains the intersection of Ω with an ϵ-neighborhood of

$$\Gamma(x_0) = \{x \in \partial\Omega, \; (x - x_0).\nu(x) \geq 0\}, \tag{7}$$

where ν is the unit outward normal vector for $\partial\Omega$ and an ϵ-neighborhood of $\Gamma(x_0)$ is defined by

$$\mathcal{N}_\epsilon(\Gamma(x_0)) = \{x \in \mathbb{R}^2 \; : \; \mathrm{dist}(x, \Gamma(x_0)) \leq \epsilon\}. \tag{8}$$

Regarding the disturbances d and e, we make the following assumptions.

$(\mathbf{H_4})$: the disturbance function $d:\mathbb{R}_+\times\Omega\longrightarrow\mathbb{R}$ belongs to $L^1(\mathbb{R}_+,L^2(\Omega))$ and satisfies the following:

$$d(t,\cdot)\in H_0^1(\Omega)\cap L^{2q}(\Omega),\ \forall t\in\mathbb{R}_+,\qquad t\mapsto\int_0^t\Delta d(s,\cdot)\,ds-d_t(t,\cdot)\in Lip\left(\mathbb{R}_+,H_0^1(\Omega)\right),\tag{9}$$

where Lip denotes the space of Lipschitz continuous functions. We also impose that the following quantities

$$C_1(d)=\int_0^\infty\int_\Omega(|d|^2+|d|^{2q})\,dx\,dt,\qquad C_2(d)=\int_0^\infty\int_\Omega|d|^m\,(d_t)^2\,dx\,dt,$$
$$C_3(d)=\int_0^\infty\int_\Omega(d_t)^2\,dx\,dt,\qquad C_4(d)=\int_0^\infty\left(\int_\Omega|d_t|^{2\left(\frac{p}{p-1}\right)}\,dx\right)^{\left(\frac{p-1}{p}\right)}\,dt,\tag{10}$$

are all finite, where p is a fixed real number so that, if $0<m\leq 2$, then $p>\frac{2}{m}$ and if $2<m<4$, then $p\in(1,\frac{m}{m-2})$.

Remark 2.1 *The fact that d belongs to $L^1(\mathbb{R}_+,L^2(\Omega))$ means that the following quantity is finite*

$$C_5(d)=\int_0^\infty||d||_{L^2(\Omega)}\,dt,\tag{11}$$

which implies that the following quantity is also finite

$$C_6(d)=\int_0^\infty\int_\Omega|d|\,dx\,dt.\tag{12}$$

$(\mathbf{H_5})$: The disturbance function $e:\mathbb{R}_+\times\Omega\longrightarrow\mathbb{R}$ belongs to $W^{1,1}(\mathbb{R}_+,L^2(\Omega))$ and satisfies the following

$$e\ \in\ Lip\left(\mathbb{R}_+,H_0^1(\Omega)\right),\quad e(0,.)\in L^2(\Omega),\quad \bar{C}_1(e)=\int_0^\infty\int_\Omega e^2\,dx\,dt<\infty.\tag{13}$$

Remark 2.2 *The fact that e belongs to $W^{1,1}(\mathbb{R}_+,L^2(\Omega))$ means that the following quantities are finite*

$$\bar{C}_2(e)=\int_0^\infty||e(t,\cdot)||_{L^2(\Omega)}\,dt,\qquad \bar{C}_3(e)=\int_0^\infty||e_t(t,\cdot)||_{L^2(\Omega)}\,dt.\tag{14}$$

Remark 2.3 *In the rest of the paper, we will use various symbols C, C_u and $C_{d,e}$ which are constants independent of the time t. However, it is important to stress that these symbols have specific dependence on other parameters of the problem. More precisely, the symbol C will be used to denote positive constants independent of initial conditions and disturbances, i.e., only depending on the domains Ω,ω and the functions a and g. The symbol C_u denotes a generic $\mathcal{K}$-function of the norms of the initial condition (u_0,u_1) and similarly the symbol $C_{d,e}$ denotes a generic $\mathcal{K}$-function of the several quantities $C_i(d)$ and $\bar{C}_i(e)$. Here $\mathcal{K}$ denotes the set of continuous increasing functions $\gamma:\ \mathbb{R}_+\to\mathbb{R}_+$ with $\gamma(0)=0$, cf. [19].*
Moreover, in the course of intermediate computations, we will try to keep all the previous constants as explicit as possible in terms of the norms of the initial condition and the $C_i(d)$ and $\bar{C}_i(e)$ in order to keep track of the nature of generic constants. We will use the latter generic mainly in the statements of the results.

Before we state the main results, we define the notion of a strong solution of ($\mathbf{P_{dis}}$). To do so, we start by giving an equivalent form of ($\mathbf{P_{dis}}$) :

Define for every $(t,x) \in \mathbb{R}_+ \times \Omega$, $\bar{d}(t,x) = \int_0^t d(s,x)ds$. We translate u in ($\mathbf{P_{dis}}$) as $v = u + \bar{d}$, it is immediate to see that ($\mathbf{P_{dis}}$) is equivalent to the following problem:

$$\begin{cases} v_{tt} - \Delta v + a(x)g(v_t) = \tilde{e}, & \text{in } \mathbb{R}_+ \times \Omega, \\ v = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ v(0,.) = v^0, \quad v_t(0,.) = \bar{u}^1, & \end{cases} \tag{15}$$

where $\tilde{e} = d_t - \Delta \bar{d} - e$, $v^0 = u^0$ and $v^1 = u^1 + d(0,.)$.
Define the unbounded operator

$$\begin{aligned} A: \; H = H_0^1(\Omega) \times L^2(\Omega) &\longrightarrow H, \\ (x_1, x_2) &\longmapsto (x_2, -\Delta x_1 + ag(x_2)), \end{aligned} \tag{16}$$

with domain

$$D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega).$$

For $t \geq 0$, set

$$U(t) = \begin{pmatrix} u(t,\cdot) \\ u_t(t,\cdot) \end{pmatrix}, \quad V(t) = \begin{pmatrix} v(t,\cdot) \\ v_t(t,\cdot) \end{pmatrix}, \quad D(t) = \begin{pmatrix} \bar{d}(t,\cdot) \\ d(t,\cdot) \end{pmatrix}, \quad G(t) = \begin{pmatrix} 0 \\ \tilde{e} \end{pmatrix}.$$

Notice that $G \in Lip(\mathbb{R}_+, L^2(\Omega) \times H_0^1(\Omega))$. Then Problem (15) can be written as

$$V_t(t) = AV(t) + G(t), \quad V(0) = V_0 \; = \begin{pmatrix} v^0 \\ v^1 \end{pmatrix}. \tag{17}$$

A strong solution of (17) in the sens of [5] is a function $V \in C(\mathbb{R}_+, H)$, absolutely continuous in every compact of $\mathbb{R}_+$, satisfying $V(t) \in D(A), \forall t \in \mathbb{R}_+$ and satisfying (17) almost everywhere in $\mathbb{R}_+$. On the other hand, the hypotheses satisfied by d imply that $D(t) \in D(A)$ for every $t \in \mathbb{R}_+$. Since $U = V - D$, we can now give the following definition for a strong solution of ($\mathbf{P_{dis}}$).

Definition 2.1 *(Strong solution of* ($\mathbf{P_{dis}}$)*.)*
A strong solution u of ($\mathbf{P_{dis}}$) is a function $u \in C^1(\mathbb{R}_+, L^2(\Omega)) \cap C(\mathbb{R}_+, H_0^1(\Omega))$ such that $t \mapsto u_t(t,\cdot)$ is absolutely continous in every compact of $\mathbb{R}_+$. For all $t \in \mathbb{R}_+$, $(u(t,\cdot), u_t(t,\cdot)) \in D(A)$ and $u(t,\cdot)$ satisfies ($\mathbf{P_{dis}}$) for almost all $t \in \mathbb{R}_+$.

We gather our findings in the following theorem regarding the disturbed system ($\mathbf{P_{dis}}$).

Theorem 2.1 *Suppose that Hypotheses ($\mathbf{H_1}$) to ($\mathbf{H_5}$) are satisfied. Then, given $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, Problem ($\mathbf{P_{dis}}$) has a unique strong solution u. Furthermore, the following energy estimate holds:*

$$E(t) \leq (C + C_u)E(0)e^{-\frac{t-1}{C_u + C}} + C_{d,e}(C_u + 1), \tag{18}$$

where the positive constant C_u depends only on the initial conditions and the positive constant $C_{d,e}$ depends only on the disturbances d and e.

Remark 2.4 ***(Comments and extensions)***

- *Theorem 2.1 holds true if the Lipschitz assumptions in* (9) *and* (13) *are replaced by bounded variation ones.*

- *In the case where the disturbances are both zero ($d \equiv 0$ and $e \equiv 0$), Theorem 2.1 holds without the hypothesis on g' given by* (5) *(i.e. no restriction on q in* (4)*) and the hypothesis given by* (4) *can be then weakened to the following one*

$$\exists\ C > 0,\ \exists\ q > 1,\ \forall\ |x| \geq 1,\ |g(x)| \leq C|x|^q.$$

It is clear that if g satisfies the last part of the condition above for $0 \leq q \leq 1$, it would still satisfy it for any $q > 1$.

- *The geometrical condition MGC imposed in* ($\mathbf{H_3}$) *can be readily reduced to the weaker and more general MGC introduced in [13] and called piecewise MGC in [3].*

Remark 2.5 *Note that* (18) *is an ISS-type estimate but it fails to be a strict one (let say in the sense of Definition* 1.6 *in [19]) for two facts. First of all, the estimated quantity E is the norm of a trajectory in the space $H_0^1(\Omega) \times L^2(\Omega)$ while the constant C_u depends on the initial condition by its norm in the smaller space $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. This difference seems unavoidable since in the undisturbed case exponential decay can be proved only for strong solutions as soon as the nonlinearity g is not assumed to be bounded below at infinity by a linear function. As a matter of fact, it would be interesting to prove that strong stability is the best convergence result one could get for weak solutions, let say with damping functions g of saturation type functions and in dimension at least two.*
The second difference lies in the second term in (18)*, namely it is not just a $\mathcal{K}$-function of the norms of the disturbances. We can get such a result if we have an extra assumption on g, typically g of growth at most linear at infinity (i.e., $q = 1$) with bounded derivative (i.e., $m = 0$). In particular, this covers the case of regular saturation functions (increasing bounded functions g with bounded derivatives).*

We give now the proof of the well-posedness part of Theorem (2.1).

Proof of the well-posedness: The argument is standard since $-A$, where A is defined in (16), is a maximal monotone operator on $H_0^1(\Omega) \times L^2(\Omega)$ (cf. for instance [11] for a proof). We can apply Theorem 3.4 combined with Propositions 3.2 and Propositions 3.3 in [5] to (17), which immediately proves the results of the well- posedness part.

■

Remark 2.6 *In [16], the domain of the operator has been chosen as*

$$\{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)\ :\ -\Delta u + g(v) \in L^2(\Omega)\}.$$

However, in dimension two, taking the domain of A in the case where $d = e = 0$ as $Z = \{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) : -\Delta u + a(x)g(v) \in L^2(\Omega)\}$ or as $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ is equivalent. Indeed, using the hypothesis given by (4)*, we have that $|g(v)| \leq C|v|^q$ for $|v| < 1$, which means when* *combining it with the fact that $g(0) = 0$ that $|g(v)| \leq C|v|^q + C|v|$ for all v. From Gagliardo-Nirenberg theorem (see in Appendix) we have for $v \in H_0^1(\Omega)$ that*

$$\|v\|^{2q}_{L^{2q}(\Omega)} \leq C\|v\|^{2q-2}_{H_0^1(\Omega)}\|v\|^2_{L^2(\Omega)},$$

which means that

$$\begin{aligned}\|g(v)\|^2_{L^2(\Omega)} = \int_\Omega |g(v)|^2 dx \leq C\int_\Omega (|v|^q + |v|)^2\, dx \leq C\|v\|^{2q}_{L^2q(\Omega)} + C\|v\|^2_{L^2(\Omega)} \\ \leq \|v\|^{2q-2}_{H_0^1(\Omega)}\|v\|^2_{L^2(\Omega)} + C\|v\|^2_{L^2(\Omega)} < +\infty \quad (\textit{since } v \in H_0^1(\Omega)),\end{aligned}$$

i.e., $g(v) \in L^2(\Omega)$. Then, by using Lemma 3.2 (with $(d,e) \equiv (0,0)$), we have that $-\Delta u + ag(v) \in L^2(\Omega)$, which means that $\Delta u \in L^2(\Omega)$. On the other hand, $\|\Delta u\|_{L^2(\Omega)}$ is an equivalent norm to the norm of $H^2(\Omega) \cap H_0^1(\Omega)$ and Ω is of class C^2 (the proof is a direct result of Theorem 4 of Section 6.3 in [9]). We can finally conclude that Z is nothing else but $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

3 Proof of the energy estimate (18)

To prove the energy estimate given by (18), we are going to use the multiplier method combined with a Gronwall lemma and other technical lemmas given in this section. We will be referring to [15] and [16] in several computations since our problem is a generalization of their strategy to the case where the disturbances (d, e) are present.

We start with the following lemma stating that the energy E is bounded along trajectories of $(\mathbf{P_{dis}})$.

Lemma 3.1 *Under the hypotheses of Theorem* (2.1)*, the energy of a strong solution of Problem* $(\mathbf{P_{dis}})$*, satisfies*

$$E'(t) = -\int_\Omega au_t g(u_t + d)\, dx - \int_\Omega u_t e\, dx, \quad \forall t \geq 0. \tag{19}$$

Furthermore, there exist positive constants C and $C_{d,e}$ such that

$$E(T) \leq CE(S) + C_{d,e}, \quad \forall\ 0 \leq S \leq T. \tag{20}$$

Proof of Lemma 3.1: Equation (19) follows after multiplying the first equation of (1) by u_t and performing standard computations. Notice that we do not have the dissipation of E since the sign of E' is not necessarily constant. To achieve (20), we first write

$$-\int_\Omega au_t g(u_t + d)\, dx = -\int_{|u_t|\leq|d|} au_t g(u_t + d)\, dx - \int_{|u_t|>|d|} au_t g(u_t + d)\, dx. \tag{21}$$

On one hand, from (3) and the fact that $(u_t + d)$ and u_t have the same sign if $|u_t| > |d|$, we deduce that

$$-\int_{|u_t|>|d|} au_t g(u_t + d)\, dx \leq 0. \tag{22}$$

On the other hand, since g is non-decreasing, has linear growth in a neighborhood of zero by (3), and satisfies (4), it follows that

$$\begin{aligned} -\int_{|u_t|\le|d|} a u_t g(u_t+d)\,dx &\le C\int_{|u_t|\le|d|} |d||g(|2d|)|\,dx \le C\int_\Omega |d||g(|2d|)|\,dx \\ &\le C\int_{|d|<1} |d||g(2d)|dx + C\int_{|d|\ge1} |d||g(2d)|dx \\ &\le C\int_{|d|<1} |d|^2 dx + C\int_{|d|\ge1} |d|^{q+1} dx \\ &\le C\int_\Omega (|d|^2+|d|^{2q})\,dx. \end{aligned} \tag{23}$$

Combining (21), (22), (23) and (19), we obtain that

$$E' \le C\int_\Omega (|d|^2+|d|^{2q})\,dx - \int_\Omega u_t e\,dx\,dt. \tag{24}$$

Using Cauchy-Schwarz inequality,

$$\begin{aligned} E' &\le C\int_\Omega (|d|^2+|d|^{2q})\,dx + \left(\int_\Omega |e|^2\,dx\,dt\right)^{\frac12}\left(\int_\Omega |u_t|^2\,dx\right)^{\frac12} \\ &\le C\int_\Omega (|d|^2+|d|^{2q})\,dx + C\|e\|_{L^2(\Omega)}\sqrt{E}, \end{aligned}$$

then integrating between two arbitrary non negative times $S \le T$, we get

$$E(T) \le E(S) + CC_1(d) + C\int_S^T \|e\|_{L^2(\Omega)}\sqrt{E}dt,$$

which allows us to apply Theorem A.2 and conclude that

$$E(T) \le CE(S) + CC_1(d) + C\bar{C}_2(e)^2 = CE(S) + C_{d,e}.$$

Hence, the proof of Lemma 3.1 is completed.

■

Remark 3.1 *In the absence of disturbances, in other words when $d = e = 0$ we have that:*

$$E'(t) = -\int_\Omega a u_t g(u_t)\,dx, \quad \forall t \ge 0, \tag{25}$$

and thus the energy E is non increasing by using (3). That latter fact simplifies the proof of exponential decrease in this case.

We provide now an extension of Lemma 2 in [16] to the context of $(\mathbf{P_{dis}})$.

Lemma 3.2 *Under the hypotheses of Theorem 2.1, for every solution of Problem $(\mathbf{P_{dis}})$ with initial conditions $(u^0, u^1) \in (H^2(\Omega)\cap H_0^1(\Omega)) \times H_0^1(\Omega)$, there exist explicit positive constants C_u and $C_{d,e}$ such that*

$$\forall t \ge 0, \quad \|-\Delta u(t,\cdot) + a(\cdot)g(u_t(t,\cdot)+d(t,\cdot)) + e(t,\cdot)\|^2_{L^2(\Omega)} + \|u_t(t,\cdot)\|^2_{H_0^1(\Omega)} \le C_u + C_{d,e}. \tag{26}$$

Proof of Lemma 3.2: We set $w := u_t$, where u is the strong solution of $(\mathbf{P_{dis}})$. We know that $w(t) \in H_0^1(\Omega)$ for every $t \geq 0$. Moreover, it is standard to show that $w(t)$ satisfies in the distributional sense the following problem:

$$\begin{cases} w_{tt} - \Delta w + ag'(w+d)(w_t + d_t) + e_t = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ w = 0, & \text{on } \partial\Omega \times \mathbb{R}_+, \\ w(0) = u^1, \quad w_t(0) = \Delta u^0 - g(u^1 + d(0)) - e(0). \end{cases} \tag{27}$$

Set $E_w(t)$ to be the energy of w for all $t \geq 0$. It is given by

$$E_w(t) = \frac{1}{2}\int_\Omega (w_t^2(t,x) + |\nabla w(t,x)|^2)\, dx.$$

Using w_t as a test function in (27), then performing standard computations, we derive

$$E_w(t) - E_w(0) = -\int_0^t \int_\Omega (ag'(w+d)(d_t + w_t)w_t + e_t w_t)\, dx d\tau. \tag{28}$$

Let $I := \int_0^t \int_\Omega a(.)g'(w+d)(d_t + w_t)w_t\, dx d\tau$. We split the domain Ω in I according to whether $|d_t| \leq |w_t|$ or not. Clearly the part corresponding to $|d_t| \leq |w_t|$ is non negative since $g' \geq 0$, $a \geq 0$ and $(d_t + w_t)$ and w_t have the same sign. From (5), one has the immediate estimate

$$g'(a+b) \leq C(1 + |a+b|^m) \leq C(1 + |a|^m + |b|^m), \quad \forall a, b \in \mathbb{R}.$$

Using the above, we can rewrite (28) as

$$\begin{aligned} E_w(t) - E_w(0) &\leq \int_0^t \int_{|d_t|>|w_t|} ag'(w+d)(d_t + w_t)w_t\, dx d\tau + \int_0^t \int_\Omega |e_t||w_t|\, dx d\tau \\ &\leq C\int_0^t \int_\Omega g'(w+d)d_t^2\, dx d\tau + C\int_0^t ||e_t||_{L^2(\Omega)}\sqrt{E_w} d\tau \\ &\leq C\int_0^t \int_\Omega (1 + |w|^m + |d|^m)d_t^2\, dx d\tau + C\int_0^t \|e_t\|_{L^2(\Omega)}\sqrt{E_w} d\tau. \end{aligned} \tag{29}$$

Using Hölder's inequality,

$$\int_0^t \int_\Omega |w|^m d_t^2\, dx d\tau \leq \int_0^t \left(\int_\Omega |w|^{pm}\, dx\right)^{\frac{1}{p}} \left(\int_\Omega |d_t|^{2p'}\, dx\right)^{\frac{1}{p'}} d\tau, \tag{30}$$

with p defined in (10) and $p' > 1$ is its conjugate exponent given by $\frac{1}{p} + \frac{1}{p'} = 1$. Thanks to the assumptions on p, one can use Gagliardo-Nirenberg's inequality for w to get

$$\left(\int_\Omega |w(t,x)|^{pm}\, dx\right)^{\frac{1}{p}} \leq CE_w(t)^{\frac{m\theta}{2}} E(t)^{\frac{(1-\theta)m}{2}}, \quad t \geq 0, \tag{31}$$

where $\theta = 1 - \frac{2}{mp}$. Combining (31), (30) and (29), it follows that

$$\begin{aligned} E_w(t) - E_w(0) \leq C\int_0^t E_w^{\frac{m\theta}{2}} E^{\frac{(1-\theta)m}{2}} \int_\Omega \left(|d_t|^{2p'}\, dx\right)^{\frac{1}{p'}} d\tau \\ + \int_0^t \int_\Omega (1 + |d|^m)d_t^2\, dx d\tau + C\int_0^t ||e_t||_{L^2(\Omega)}\sqrt{E_w} d\tau. \end{aligned} \tag{32}$$

Note that $\frac{m\theta}{2} < 1$. Setting $h_1(t) = \int_\Omega \left(|d_t|^{2p'}\, dx\right)^{\frac{1}{p'}}$, $h_2(t) = ||e_t||_{L^2(\Omega)}$ and using (20), (32) becomes

$$E_w(t) \le E_w(0) + C_2(d) + C_3(d) + (C_u + C_{d,e}) \int_0^t E_w^{\frac{m\theta}{2}} h_1(s) ds + C \int_0^t h_2(s) \sqrt{E_w} ds. \tag{33}$$

We know that

$$\int_0^\infty h_1(t)\, dt = C_4(d) < \infty, \quad \int_0^\infty h_2(t)\, dt = \bar{C}_3(e) < \infty. \tag{34}$$

We can now apply Theorem A.2 on (33) with

$$S = 0,\ T = t,\ \alpha_1 = \frac{m\theta}{2},\ \alpha_1 = \frac{1}{2},\ F(\cdot) = E_w(\cdot),\ C_3 = C_2(d) + C_3(d),\ C_1 = C_u + C_{d,e},\ C_2 = C.$$

We obtain the following bound for $E_w(\cdot)$:

$$E_w(t) \le \max\left(2(E_w(0) + C_2(d) + C_3(d)), (2\tilde{C})^{\frac{1}{1-\alpha}}\right), \tag{35}$$

where $\tilde{C} := C_1\|h_1\|_1 + C_2\|h_2\|_1$ and $\alpha := \max(\alpha_1, \alpha_2)$ if $2\tilde{C} \ge 1$ or $\alpha := \min(\alpha_1, \alpha_2)$ if $2\tilde{C} < 1$.

It is clear that $\tilde{C} = (C_u + C_{d,e})C_4(d) + C\bar{C}_3(e) \le C_u + C_{d,e}$. One then rewrites (35) as

$$E_w(t) \le 2(E_w(0) + C_2(d) + C_3(d)) + (C_u + C_{d,e})^{\frac{1}{1-\alpha}}. \tag{36}$$

Note that for $t \ge 0$ one obviously has that

$$\begin{aligned} E_w(t) &= \frac{1}{2}\int_\Omega (w_t^2(t,x) + |\nabla w(t,x)|^2)\, dx \\ &= \frac{1}{2}\left(||u_{tt}(t,\cdot)||^2_{L^2(\Omega)} + \|u_t(t,\cdot)\|^2_{H_0^1(\Omega)}\right) \\ &= \| - \Delta u(t,\cdot) + a(\cdot)g(u_t(t,\cdot) + d(t,\cdot)) + e(t,\cdot)\|^2_{L^2(\Omega)} + \|u_t(t,\cdot)\|^2_{H_0^1(\Omega)}. \end{aligned}$$

The conclusion of the lemma follows since, by taking into account (4), it is clear that $E_w(0) \le C_u + C_{d,e}$.

■

We next provide the following important estimate based on Gagliardo-Nirenberg theorem:

Lemma 3.3 *For all $q > 2$, a strong solution u of $(\mathbf{P_{dis}})$ satisfies*

$$\|u_t(t,\cdot)\|^q_{L^q(\Omega)} \le (C_u + C_{d,e})E(t), \quad t \ge 0. \tag{37}$$

Proof of Lemma 3.3: We derive immediately from (26) that $\|u_t\|_{H_0^1(\Omega)} \le C_u + C_{d,e}$. Then, using Gagliardo-Nirenberg's theorem, it follows that, for every $t \ge 0$,

$$\|u_t(t,\cdot)\|^q_{L^q(\Omega)} \le C\|u_t(t,\cdot)\|^{q-2}_{H_0^1(\Omega)}\|u_t(t,\cdot)\|^2_{L^2(\Omega)} \le (C_u + C_{d,e})E(t). \tag{38}$$

■

We have all the tools now to start the proof of the second part of Theorem 2.1. The stability result will be achieved as a direct consequence of the following proposition:

Proposition 3.1 *Suppose that the hypotheses of Theorem* (2.1) *are satisfied, then the energy* E *of the strong solution* u *of* $(\mathbf{P_{dis}})$ *with* $(u^0,u^1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega))$, *satisfies the following estimate:*

$$\int_S^T E(t)\,dt \leq (C_u+C)E(S)+(1+C_u)C_{d,e}, \tag{39}$$

where the positive constant C_u *depends only on the initial condition, the positive constant* $C_{d,e}$ *depends only on the disturbances* d *and* e *respectively and* C *is a positive real constant.*

3.0.1 Proof of Proposition 3.1

We now embark on an argument for Proposition 3.1. It is based on the use of several multipliers that we will apply to the partial differential equation of (1). For that purpose, we need to define several functions associated with Ω.

Let $(u^0,u^1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$, $S\leq T$ two non negative times and $x_0\in\mathbb{R}^2$ an observation point. Define ϵ_0, ϵ_1 and ϵ_2 three positive real constants such that $\epsilon_0<\epsilon_1<\epsilon_2<\epsilon$ where ϵ is the same defined in 8. Using ϵ_i, we define Q_i for $i=0,1,2$ as $Q_i=\mathcal{N}_{\epsilon_i}[\Gamma(x_0)]$.
Since $\overline{(\Omega\setminus Q_1)}\cap\overline{Q_0}=\emptyset$, we are allowed to define a function $\psi\in C_0^\infty(\mathbb{R}^2)$ such that

$$\begin{cases} 0\leq\psi\leq 1, \\ \psi=1 \text{ on } \bar{\Omega}\setminus Q_1, \\ \psi=0 \text{ on } Q_0. \end{cases}$$

We also define the C^1 vector field h on Ω by

$$h(x):=\psi(x)(x-x_0). \tag{40}$$

When the context is clear, we will omit the arguments of h.

We use the multiplier $M(u):=h\nabla u+\frac{u}{2}$ to deduce the following first estimate:

Lemma 3.4 *Under the hypotheses of Proposition 3.1, we have the following inequality:*

$$\int_S^T E\ dt \leq \underbrace{\left|\left[\int_\Omega u_t M(u)\,dx\right]_S^T\right|}_{\mathbf{T_1}} + \underbrace{C\int_S^T\int_{\Omega\cap Q_1}|\nabla u|^2\,dx\,dt}_{\mathbf{T_2}} + \underbrace{\left|\int_S^T\int_\Omega ag(u_t+d)M(u)\,dx\,dt\right|}_{\mathbf{T_3}}$$
$$+\underbrace{\left|\int_S^T\int_\Omega eM(u)\,dx\,dt\right|}_{\mathbf{T_4}} + \underbrace{C\int_S^T\int_\omega u_t^2\,dx\,dt}_{\mathbf{T_5}}, \tag{41}$$

where h *is defined in* (40) *and* $M(u)$ *is the multiplier given by* $h.\nabla u+\frac{u}{2}$.

Proof of Lemma 3.4. The proof is based on multiplying $(\mathbf{P_{dis}})$ by the multiplier $M(u)$ and integrating on $[S,T]\times\Omega$. Then, we follow the steps that led to the proof of equation (3.15) in [15] except that we take $\sigma=0$ and $\phi(t)=t$ in the beginning and we replace $\rho(x,u_t)$ by $a(x)g(u_t+d)+e$.

■

Remark 3.2 *From now on, whenever we refer to a proof in [15], we refer to the steps of the proof with the change of $\sigma = 0$ and $\phi(t) = t$ as well as replacing $\rho(x, u_t)$ by $a(x)g(u_t + d) + e$.*

The goal now is to estimate the terms T_1 to T_5.

Lemma 3.5 *Under the hypotheses of Proposition 3.1, there exists a positive constant C such that*

$$T_1 \leq CE(S) + C_{d,e}. \tag{42}$$

Proof of Lemma 3.5: Exactly as the proof of equation (5.14) in [15] except that we use (20) in the very last step since we do not have the non-increasing of the energy here. We obtain (42).

■

The estimation of T_2 requires more work and it is given in the following lemma:

Lemma 3.6 *Under the hypotheses of Proposition 3.1, T_2 is estimated by*

$$T_2 \leq C\eta_0 \int_S^T E\ dt + \frac{C}{\eta_0} \int_S^T \int_\omega u_t^2\, dx\, dt + \frac{1}{\eta_0} \left(C + C_u + C_{d,e}\right) E(S) + \frac{1}{\eta_0^5} \left(C_{d,e} C_u + C_{d,e}\right), \tag{43}$$

where $0 < \eta_0 < 1$ is an arbitrary real positive number to be chosen later.

Proof of Lemma 3.6: The argument requires a new multiplier, namely ξu, where the function $\xi \in C_0^\infty(\mathbb{R}^2)$ is defined by

$$\begin{cases} 0 \leq \xi \leq 1, \\ \xi = 1 \text{ on } Q_1, \\ \xi = 0 \text{ on } \mathbb{R}^2 \setminus Q_2. \end{cases} \tag{44}$$

Such a function ξ exists since $\overline{\mathbb{R}^2 \setminus Q_2} \cap \overline{Q_1} = \emptyset$. Using the multiplier ξu and following the steps in the proof of Lemma 9 in [15], yields the following identity:

$$\int_S^T \int_\Omega \xi |\nabla u|^2\, dx\, dt = \int_S^T \int_\Omega \xi |u_t|^2\, dx\, dt + \frac{1}{2} \int_S^T \int_\Omega \Delta\xi u^2\, dx\, dt - \left[\int_\Omega \xi u u_t\, dx \right]_S^T \\ - \int_S^T \int_\Omega \xi u \left[a(x) g(u_t + d) + e \right]\ dx\, dt. \tag{45}$$

Combining the fact that $\Delta\xi$ is bounded and the definition of ξ, we derive from (45) that

$$T_2 \leq \int_S^T \int_{\Omega \cap Q_2} |u_t|^2\, dx\, dt + \underbrace{\left| \left[\int_{\Omega \cap Q_2} u u_t\, dx \right]_S^T \right|}_{S_1} + \underbrace{C \int_S^T \int_{\Omega \cap Q_2} u^2\, dx\, dt}_{S_2} \\ + \underbrace{\int_S^T \int_\Omega |u a g(u_t + d)|\, dx\, dt}_{S_3} + \underbrace{\int_S^T \int_\Omega |u e|\, dx\, dt}_{S_4}. \tag{46}$$

First, note that the first term of (46) is upper bounded by $\int_S^T \int_\omega |u_t|^2\,dx\,dt$ since $\Omega \cap Q_2 \subset \omega$. Left to estimate the other terms in the right-hand side of (46). We start by treating S_1. We easily get the following estimate by using Young and Poincaré inequalities:

$$\int_{\Omega\cap Q_2} |uu_t|\,dx \leq \frac{1}{2}\int_{\Omega\cap Q_2} |u|^2\,dx + \frac{1}{2}\int_{\Omega\cap Q_2} |u_t|^2\,dx \leq CE. \tag{47}$$

Using (20) with (47) we obtain the estimation of S_1 given by

$$S_1 \leq CE(S) + C_{d,e}. \tag{48}$$

To estimate S_2, we introduce the last multiplier in what follows:

Since $(\overline{\Omega \setminus \omega}) \cap (\overline{Q_2 \cap \Omega}) = \emptyset$, there exists a function $\beta \in C_0^\infty(\mathbb{R}^2)$ such that

$$\begin{cases} 0 \leq \beta \leq 1, \\ \beta = 1 \quad \text{on} \quad Q_2 \cap \Omega, \\ \beta = 0 \quad \text{on} \quad \Omega \setminus \omega. \end{cases} \tag{49}$$

For every $t \geq 0$, let z be the solution of the following elliptic problem:

$$\begin{cases} \Delta z = \beta u & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{50}$$

One can prove the following lemma:

Lemma 3.7 *Under the hypotheses of Proposition 3.1 with z as defined in* (50)*, it holds that*

$$||z||_{L^2(\Omega)} \leq C||u||_{L^2(\Omega)}, \quad ||z_t||^2_{L^2(\Omega)} \leq C\int_\Omega \beta|u_t|^2\,dx, \quad \|\nabla z\|_{L^2(\Omega)} \leq C||\nabla u||_{L^2(\Omega)}, \tag{51}$$

$$\forall S \leq T \in \mathbb{R}_+, \quad \int_S^T \int_\Omega \beta u^2\,dx\,dt = \left[\int_\Omega z u_t\,dx\right]_S^T + \int_S^T \int_\Omega (-z_t u_t + z\,[ag(u_t+d)+e])\,dx\,dt. \tag{52}$$

Proof of Lemma 3.7: Equation 51 gathers standard elliptic estimates from the definition of z as a solution of (50) while (52) is obtained by using z as a multiplier for ($\mathbf{P_{dis}}$). Steps of the proof are similar to the ones that led to equations (5.22), (5.25) and (5.26) in [15].

■

Since the non negative β is equal to 1 on Q_2 and 0 on $\mathbb{R}^2 \setminus \omega$, it follows from (52) that

$$S_2 \leq \underbrace{\left[\int_\Omega z u_t\,dx\right]_S^T}_{U_1} - \underbrace{\int_S^T \int_\Omega z_t u_t \;\, dx\,dt}_{U_2} + \underbrace{\int_S^T \int_\Omega z(ag(u_t+d)+e) \;\, dx\,dt}_{U_3}. \tag{53}$$

We estimate U_1, U_2 and U_3. We start by handling U_1. One has from Cauchy-Schwarz inequality, then (51) and Poincaré inequality that

$$\left|\int_\Omega z u_t\,dx\right| \leq ||z||_{L^2(\Omega)}||u_t||_{L^2(\Omega)} \leq C||\nabla u||_{L^2(\Omega)}||u_t||_{L^2(\Omega)} \leq CE(t). \tag{54}$$

Using (54) and the fact that E is non-increasing, it is then immediate to derive that

$$|U_1| = \left| \left(\int_\Omega z u_t \, dx \right)(T) - \left(\int_\Omega z u_t \, dx \right)(S) \right| \leq C(E(T) + E(S)). \tag{55}$$

Finally, using (20) in (55), we obtain that

$$U_1 \leq CE(S) + C_{d,e}. \tag{56}$$

As for U_2, the use of Young inequality with an arbitrary real number $0 < \eta_0 < 1$ yields

$$|U_2| \leq \int_S^T \int_\Omega \frac{1}{2\eta_0} |z_t|^2 \, dx \, dt + \int_S^T \int_\Omega \frac{\eta_0}{2} |u_t|^2 \, dx \, dt.$$

Then, we use (51) and the fact that $0 \leq \beta \leq 1$ to conclude the following estimate:

$$U_2 \leq \frac{C}{\eta_0} \int_S^T \int_\omega u_t^2 \, dx \, dt + C\eta_0 \int_S^T E \, dx \, dt, \tag{57}$$

where η_0 is a positive real number to be chosen later.
Left to estimate U_3. We can rewrite it as the following:

$$U_3 = \underbrace{\int_S^T \int_{|u_t+d|\leq 1} a(x) z g(u_t + d) dx dt}_{V_1} + \underbrace{\int_S^T \int_{|u_t+d|>1} a(x) z g(u_t + d) dx dt}_{V_2} + \underbrace{\int_S^T \int_\Omega a(x) z e dx dt}_{V_3}. \tag{58}$$

We estimate the three terms V_1, V_2 and V_3. We start by estimating V_1. We have using Young inequality that

$$V_1 \leq C\eta_0 \int_S^T E \, dt + \frac{1}{\eta_0} \int_S^T \int_{|u_t+d|\leq 1} |a g(u_t + d)|^2 \, dx \, dt. \tag{59}$$

The fact that $g(0) = 0$ implies the existence of a constant $C > 0$ such that $|g(x)| \leq C|x|$ for all $|x| \leq 1$. Combining it with the fact that $g(x)x \geq 0, \ \forall \, x \in \mathbb{R}$, it follows that

$$\begin{aligned} \int_S^T \int_{|u_t+d|\leq 1} |a g(u_t + d)|^2 \, dx \, dt &\leq \int_S^T \int_{|u_t+d|\leq 1} a(.)(u_t + d) g(u_t + d) \, dx \, dt \\ &\leq \int_S^T \int_\Omega a(.)(u_t + d) g(u_t + d) \, dx \, dt. \end{aligned} \tag{60}$$

Using (19) and Young inequality with $0 < \eta_1 < 1$,

$$\begin{aligned} &\int_S^T \int_\Omega a(.)(u_t + d) g(u_t + d) \, dx \, dt = \int_S^T \int_\Omega a u_t g(u_t + d) \, dx \, dt + \int_S^T \int_\Omega a d g(u_t + d) \, dx \, dt \\ &\leq \int_S^T \int_\Omega a u_t g(u_t + d) \, dx \, dt + \int_S^T \int_\Omega u_t e \, dx \, dt - \int_S^T \int_\Omega u_t e \, dx \, dt + C \int_S^T \int_\Omega |d||g(u_t + d)| \, dx \, dt \\ &\qquad \leq \int_S^T (-E') dt + \int_S^T \int_\Omega |u_t||e| \, dx \, dt + C \int_S^T \int_\Omega |d||g(u_t + d)| \, dx \, dt \\ &\qquad \leq E(S) + C\eta_1 \int_S^T E \, dt + \frac{C}{\eta_1} \int_S^T \int_\Omega |e|^2 \, dx \, dt + C \int_S^T \int_\Omega |d||g(u_t + d)| \, dx \, dt \\ &\qquad \leq E(S) + C\eta_1 \int_S^T E \, dt + \frac{C}{\eta_1} \bar{C}_1(e) + C \int_S^T \int_\Omega |d||g(u_t + d)| \, dx \, dt. \end{aligned} \tag{61}$$

Left to estimate $\int_S^T \int_\Omega |d||g(u_t+d)|\,dx\,dt$, we proceed as the following:

$$\begin{aligned}
\int_S^T \int_\Omega |d||g(u_t+d)|\,dx\,dt &= \int_S^T \int_{|u_t+d|\le 1} |d||g(u_t+d)|\,dx\,dt + \int_S^T \int_{|u_t+d|>1} |d||g(u_t+d)|\,dx\,dt \\
&\le C\int_S^T \int_{|u_t+d|\le 1} |d|\,dx\,dt + \frac{C}{\eta_1'}\int_S^T \int_{|u_t+d|>1} |d|^2\,dx\,dt + \eta_1'\int_S^T \int_{|u_t+d|>1} |g(u_t+d)|^2\,dx\,dt \\
&\le CC_6(d) + \frac{C}{\eta_1'}C_1(d) + C\eta_1'\int_S^T \int_{|u_t+d|>1} |u_t+d|^{2q}\,dx\,dt \\
&\le CC_6(d) + \frac{C}{\eta_1'}C_1(d) + C\eta_1'\int_S^T \int_\Omega |u_t|^{2q} + C\eta_1'\int_S^T \int_\Omega |d|^{2q}\,dx\,dt, \qquad (62)
\end{aligned}$$

where $0<\eta_1'<1$. Then, using (37),

$$\begin{aligned}
\int_S^T \int_\Omega |d||g(u_t+d)|\,dx\,dt &\le CC_6(d) + \frac{C}{\eta_1'}C_1(d) + \eta_1'(C_u+C_{d,e})\int_S^T E(t)\,dt + C\eta_1'C_1(d) \\
&\le \frac{1}{\eta_1'}C_{d,e} + \eta_1'(C_u+C_{d,e})\int_S^T E(t)\,dt. \qquad (63)
\end{aligned}$$

Combining (61) and (63),

$$\int_S^T \int_\Omega a(.)(u_t+d)g(u_t+d)\,dx\,dt \le E(S) + (\eta_1+\eta_1'(C_u+C_{d,e}))\int_S^T E\ dt + \frac{1}{\eta_1\eta_1'}C_{d,e}. \qquad (64)$$

Combining now (64), (61) and (59), we obtain that

$$V_1 \le C\left(\eta_0 + \frac{\eta_1'}{\eta_0}(C_u+C_{d,e}) + \frac{\eta_1}{\eta_0}\right)\int_S^T E\ dt + \frac{C}{\eta_0}E(S) + \frac{1}{\eta_1\eta_0\eta_1'}C_{d,e}.$$

We take $\eta_1=\eta_0^2$ and $\eta_1' = \frac{\eta_0^2}{C_u+C_{d,e}}$ if $C_u+C_{d,e}>0$. In that case, V_1 would be estimated by

$$V_1 \le C\eta_0\int_S^T E\ dt + \frac{C}{\eta_0}E(S) + \frac{1}{\eta_0^5}C_{d,e}(C_u+C_{d,e}). \qquad (65)$$

If $C_u=C_{d,e}=0$, the above equation holds true trivially.

Remark 3.3 *With such a choice of η_1 and η_1', we have the following useful estimate obtained from* (64)*:*

$$\int_S^T \int_\Omega a(.)(u_t+d)g(u_t+d)\,dx\,dt \le E(S) + C\eta_0^2\int_S^T E\ dt + \frac{1}{\eta_0^4}(C_{d,e}C_u + C_{d,e}). \qquad (66)$$

To estimate V_2, first notice that from Rellich-Kondrachov's theorem in dimension two (cf. [6]) that $H^1(\Omega)\subset L^{q+1}(\Omega)$, which means that $\exists\ C>0$ such that $\|z\|_{L^{q+1}(\Omega)} \le C\|z\|_{H^1(\Omega)}$, adding to that the fact that $z\in H_0^1(\Omega)$ and (51), it holds that

$$\|z\|_{L^{q+1}(\Omega)} \le C\sqrt{E}. \qquad (67)$$

Then, using Hölder inequality yields

$$V_2 \le \int_S^T \left(\int_{|u_t+d|>1} (a|g(u_t+d)|)^{\frac{q+1}{q}} \, dx \right)^{\frac{q}{q+1}} \left(\int_{|u_t+d|>1} |z|^{q+1} \, dx \right)^{\frac{1}{q+1}} dt. \tag{68}$$

Combining (68) with the hypothesis given by (4), we get that

$$V_2 \le C \int_S^T \left(\int_{|u_t+d|>1} a|u_t+d||g(u_t+d)| \, dx \right)^{\frac{q}{q+1}} \left(\int_{|u_t+d|>1} |z|^{q+1} \, dx \right)^{\frac{1}{q+1}} dt.$$

Using Young inequality for an arbitrary $0 < \eta_2 < 1$,

$$\begin{aligned} V_2 &\le C \int_S^T \left(\frac{1}{\eta_2^{\frac{q+1}{q}}} \int_{|u_t+d|>1} a(x)(u_t+d)g(u_t+d) \, dx + \eta_2^{q+1} \int_\Omega |z|^{q+1} \, dx \right) dt \\ &\le C \int_S^T \left(\frac{1}{\eta_2^{\frac{q+1}{q}}} \int_\Omega a(x)(u_t+d)g(u_t+d) \, dx + \eta_2^{q+1} \int_\Omega |z|^{q+1} \, dx \right) dt \\ &\le C \int_S^T \left(\frac{1}{\eta_2^{\frac{q+1}{q}}} \int_\Omega a(x)u_t g(u_t+d) \, dx + \frac{C}{\eta_2^{\frac{q+1}{q}}} \int_\Omega |d||g(u_t+d)| \, dx + \eta_2^{q+1} \int_\Omega |z|^{q+1} \, dx \right) dt. \end{aligned} \tag{69}$$

The previous inequality combined with (19) and (67) implies that

$$V_2 \le C \int_S^T \left(\frac{1}{\eta_2^{\frac{q+1}{q}}}(-E') - \frac{1}{\eta_2^{\frac{q+1}{q}}} \int_\Omega u_t e \, dx + \frac{C}{\eta_2^{\frac{q+1}{q}}} \int_\Omega |d||g(u_t+d)| \, dx + \eta_2^{q+1} E^{\frac{q+1}{2}} \right) dt.$$

Then, using (20), E satisfies

$$\int_S^T E^{\frac{q+1}{2}} \, dt = \int_S^T E^{\frac{q-1}{2}} E \, dt \le (CE(0) + C_{d,e})^{\frac{q-1}{2}} \int_S^T E \, dt \le (C_u + C_{d,e}) \int_S^T E \, dt, \tag{70}$$

which gives that

$$V_2 \le \frac{C}{\eta_2^{\frac{q+1}{q}}} E(S) + \eta_2^{q+1}(C_u + C_{d,e}) \int_S^T E dt + \frac{C}{\eta_2^{\frac{q+1}{q}}} \int_S^T \left(-\int_\Omega u_t e \, dx + \int_\Omega |d||g(u_t+d)| \, dx \right) dt.$$

We fix $\eta_2 = \left(\frac{\eta_0}{(C_u + C_{d,e})} \right)^{\frac{1}{q+1}}$. It follows that

$$\eta_2^{q+1}(C_u + C_{d,e}) = \eta_0,$$

$$\frac{C}{\eta_2^{\frac{q+1}{q}}} = C \frac{(C_u + C_{d,e})^{\frac{1}{q}}}{\eta_0^{\frac{1}{q}}} \le \frac{C}{\eta_0}(C_u^{\frac{1}{q}} + C_{d,e}^{\frac{1}{q}}) = \frac{1}{\eta_0^{\frac{1}{q}}} (C_u + C_{d,e}),$$

which leads to

$$V_2 \le \frac{1}{\eta_0^{\frac{1}{q}}}(C_u + C_{d,e})E(S) + \eta_0 \int_S^T E dt + \frac{1}{\eta_0^{\frac{1}{q}}}(C_u + C_{d,e}) \int_S^T \left(-\int_\Omega u_t e \, dx + \int_\Omega |d||g(u_t+d)| \, dx \right) dt. \tag{71}$$

To finish the estimation of V_2, we still have to handle the last two integral terms in (71). On one hand, we have already estimated the term $\int_S^T \int_\Omega |d||g(u_t+d)|\,dx\,dt$ in (63). We have immediately for some $0<\eta_3<1$ that

$$(C_u + C_{d,e}) \int_S^T \int_\Omega |d||g(u_t+d)|\,dx\,dt \le \eta_3 (C_u + C_{d,e}) \int_S^T E\ dt + \frac{1}{\eta_3}(C_{d,e}C_u + C_{d,e}). \tag{72}$$

Choosing η_3 to be equal to $\frac{\eta_0^{\frac{q+1}{q}}}{(C_u+C_{d,e})}$ implies that

$$\eta_3(C_u + C_{d,e}) = \eta_0^{\frac{q+1}{q}},$$
$$\frac{1}{\eta_3}(C_{d,e}C_u + C_{d,e}) \le \frac{1}{\eta_0^{\frac{q+1}{q}}}(C_{d,e}C_u + C_{d,e}),$$

which gives that

$$(C_u + C_{d,e}) \int_S^T \int_\Omega |d||g(u_t+d)|\,dx\,dt \le \eta_0^{\frac{q+1}{q}} \int_S^T E\ dt + \frac{1}{\eta_0^{\frac{q+1}{q}}}\left(C_{d,e}C_u + C_{d,e}\right). \tag{73}$$

On the other hand, we have for $0<\eta_4<1$ that

$$(C_u + C_{d,e}) \int_S^T \int_\Omega u_t e\,dx\,dt \le \eta_4(C_u + C_{d,e}) \int_S^T E\ dt + \frac{1}{\eta_4}(C_{d,e}C_u + C_{d,e}).$$

Using the same concept as before, we fix $\eta_4 = \frac{\eta_0^{\frac{q+1}{q}}}{C_u+C_{d,e}}$, we obtain that

$$(C_u + C_{d,e}) \int_S^T \int_\Omega u_t e\,dx\,dt \le \eta_0^{\frac{q+1}{q}} \int_S^T E\ dt + \frac{1}{\eta_0^{\frac{q+1}{q}}}\left(C_{d,e}C_u + C_{d,e}\right), \tag{74}$$

Combining (71), (73) and (74), we conclude that the estimation of V_2 is given by

$$V_2 \le \frac{1}{\eta_0^{\frac{1}{q}}}(C_u + C_{d,e})E(S) + \eta_0 \int_S^T E dt + \frac{1}{\eta_0^{\frac{q+2}{q}}}\left(C_{d,e}C_u + C_{d,e}\right). \tag{75}$$

As for V_3, we simply have when using (51) and Young inequality with η_0 that

$$V_3 \le C\eta_0 \int_S^T E\ dt + \frac{C}{\eta_0}\bar{C}_1(e),$$

which means that

$$V_3 \le C\eta_0 \int_S^T E\ dt + \frac{1}{\eta_0}C_{d,e}. \tag{76}$$

To achieve an estimation of S_2, we just combine (56),(57) (65), (75) and (76) to get

$$S_2 \le C\eta_0 \int_S^T E\ dt + \frac{C}{\eta_0}\int_S^T \int_\omega u_t^2\,dx\,dt + \left(C + \frac{C}{\eta_0} + \frac{1}{\eta_0^{\frac{1}{q}}}(C_u + C_{d,e})\right)E(S)$$
$$+ \left(\frac{1}{\eta_0^5} + \frac{1}{\eta_0^{\frac{q+2}{q}}}\right)(C_{d,e}C_u + C_{d,e}) + \left(\frac{1}{\eta_0} + 1\right)C_{d,e}. \tag{77}$$

We can simplify the previous estimate by using the fact that $0 < \eta_0 < 1$. As a result, (77) becomes

$$S_2 \le C\eta_0 \int_S^T E \; dt + \frac{C}{\eta_0} \int_S^T \int_\omega u_t^2 \, dx \, dt + \frac{1}{\eta_0} \left(C + C_u + C_{d,e} \right) E(S) + \frac{1}{\eta_0^5} \left(C_{d,e} C_u + C_{d,e} \right). \tag{78}$$

Regarding S_3, we follow the same steps we followed to get $V_1 + V_2$. It is possible because u satisfies the same result (67) as z from before. Hence, we obtain that

$$S_3 \le C\eta_0 \int_S^T E \; dt + \frac{1}{\eta_0} \left(C + C_u + C_{d,e} \right) E(S) + \frac{1}{\eta_0^5} \left(C_{d,e} C_u + C_{d,e} \right). \tag{79}$$

Finally, to estimate S_4, we simply have when using young inequality that

$$S_4 \le \eta_0 \int_S^T E \; dt + \frac{1}{\eta_0} C_{d,e}. \tag{80}$$

We complete the estimate of T_2 in (46) by combining the estimations of S_1, S_2, S_3 and S_4. Hence the proof of Lemma 3.6 is completed.

■

An estimate of T_3 is provided in the next lemma:

Lemma 3.8 *Under the hypotheses of Proposition 3.1, we have the following estimate:*

$$T_3 \le C\eta_0 \int_S^T E \; dt + \frac{1}{\eta_0} \left[C + (1 + C_{\eta_0})(C_u + C_{d,e}) \right] E(S) + \frac{1}{\eta_0^5} \left(C_{\eta_0}^3 + 1 \right) \left(C_{d,e} C_u + C_{d,e} \right), \tag{81}$$

where $0 < \eta_0 < 1$ is a positive arbitrary real number to be chosen later and C_{η_0} is an implicit positive constant that depends on η_0 only.

Proof of Lemma 3.8: First, note that

$$T_3 \le \frac{1}{2} S_3 + \underbrace{\int_S^T \int_\Omega \left| a g(u_t + d) \nabla u . h \right| \, dx \, dt}_{X}. \tag{82}$$

We have already estimated S_3 in (79). It remains to deal with X. Using Young inequality implies that

$$\begin{aligned} X &\le \frac{C}{\eta_0} \int_S^T \int_\Omega (a|g(u_t + d)|)^2 \, dx \, dt + C\eta_0 \int_S^T \int_\Omega |\nabla u|^2 \, dx \, dt \\ &\le \frac{C}{\eta_0} \int_S^T \int_\Omega a|g(u_t + d)|^2 \, dx \, dt + C\eta_0 \int_S^T E \; dt. \end{aligned} \tag{83}$$

Now, set $R_1 > 1$ to be chosen later. We can rewrite the term $\int_S^T \int_\Omega a|g(u_t + d)|^2 \, dx \, dt$ as

$$\int_S^T \int_\Omega a|g(u_t + d)|^2 \, dx \, dt = \underbrace{\int_S^T \int_{|u_t + d| \le R_1} a|g(u_t + d)|^2 \, dx \, dt}_{Y_1} + \underbrace{\int_S^T \int_{|u_t + d| > R_1} a|g(u_t + d)|^2 \, dx \, dt}_{Y_2}. \tag{84}$$

Since $g(0)=0$, it holds that $|g(x)| \le C_{R_1}|x|$ for some constant C_{R_1} and for $|x|<R_1$. Combine it with (66), it follows that Y_1 satisfies for some $0<\eta_5<1$

$$\begin{aligned} Y_1 &\le C_{R_1}\int_S^T \int_{|u_t+d|\le R_1} |ag(u_t+d)||u_t+d|\,dx\,dt \\ &\le C_{R_1}\int_S^T \int_\Omega |ag(u_t+d)||u_t+d|\,dx\,dt \\ &\le C_{R_1}E(S) + CC_{R_1}\eta_5^2\int_S^T E\ dt + \frac{C_{R_1}}{\eta_5^4}(C_{d,e}C_u + C_{d,e}). \end{aligned} \tag{85}$$

Taking $\eta_5 = \frac{\eta_0}{\sqrt{CC_{R_1}}}$ leads to

$$Y_1 \le C_{R_1}E(S) + \eta_0^2\int_S^T E\ dt + \frac{C_{R_1}^3}{\eta_0^4}(C_{d,e}C_u + C_{d,e}) \tag{86}$$

As for Y_2, we use (4) to obtain that

$$\begin{aligned} Y_2 &\le C\int_S^T \int_{|u_t+d|>R_1} |u_t+d|^{2q}\,dx\,dt \\ &\le C\int_S^T \int_{|u_t+d|>R_1} |u_t|^{2q}\,dx\,dt + C\int_S^T \int_{|u_t+d|>R_1} |d|^{2q}\,dx \\ &\le C\int_S^T \int_{|u_t+d|>R_1} \frac{|u_t+d|}{R_1}|u_t|^{2q}\,dx\,dt + C\int_S^T \int_\Omega |d|^{2q}\,dx\,dt \\ &\le C\int_S^T \int_\Omega \frac{|u_t|}{R_1}|u_t|^{2q}\,dx\,dt + C\int_S^T \int_\Omega \frac{|d|}{R_1}|u_t|^{2q}\,dx\,dt + CC_1(d) \\ &\le \frac{C}{R_1}\int_S^T \int_\Omega |u_t|^{2q+1}\,dx\,dt + \frac{C}{R_1^2}\int_S^T \int_\Omega |u_t|^{4q}\,dx\,dt + C_{d,e}. \end{aligned}$$

Then, we use Lemma 3.3 as well as the fact that $R_1>1$ to conclude that Y_2 satisfies

$$Y_2 \le \frac{1}{R_1}(C_u + C_{d,e})\int_S^T E\ dt + C_{d,e}.$$

We take $R_1 = \frac{(C_u+C_{d,e})}{\eta_0^2}$, we get the simplified estimate

$$Y_2 \le \eta_0^2\int_S^T E\ dt + C_{d,e}. \tag{87}$$

Remark 3.4 *For such a choice of R_1, and based on how C_{R_1} is defined, we can assume that C_{R_1} in* (86) *is a constant of the type $C_{\eta_0}(C_u+C_{d,e})$, where C_{η_0} is a positive constant that depends on η_0 only.*

Combining (83), (84), (86) and (87) implies that

$$X \le C\eta_0\int_S^T E\ dt + \frac{C_{\eta_0}}{\eta_0}(C_{d,e}+C_u)E(S) + \frac{C_{\eta_0}^3}{\eta_0^5}(C_{d,e}C_u + C_{d,e}) + \frac{C_{d,e}}{\eta_0}. \tag{88}$$

Finally, we combine (82) and (88) with the estimation of S_3, we obtain (81).

■

We next seek to prove the upper bound of T_4 that is given by the following lemma

Lemma 3.9 *Under the hypotheses of Proposition 3.1, the following estimate holds:*

$$T_4 \leq C\eta_0 \int_S^T E\ dt + \frac{C}{\eta_0} C_{d,e}, \tag{89}$$

where $0 < \eta_0 < 1$ *is a positive constant to be chosen later.*

***Proof of Lemma 3.9*:** We have that

$$T_4 \leq \frac{1}{2} \int_S^T \int_\Omega |eu|\, dx\, dt + \int_S^T \int_\Omega |e\nabla u.h|\, dx\, dt. \tag{90}$$

On one hand, using Young inequality gives that

$$\int_S^T \int_\Omega |eu|\, dx\, dt \leq \eta_0 \int_S^T E\ dt + \frac{C}{\eta_0} C_{d,e}. \tag{91}$$

On the other hand, it gives that

$$\int_S^T \int_\Omega |e\nabla u.h|\, dx\, dt \leq \eta_0 \int_S^T E\ dt + \frac{C}{\eta_0} C_{d,e} \tag{92}$$

Combining (90), (91) and (92), we prove (89).

■

It remains to handle the last term T_5.

Lemma 3.10 *Under the hypotheses of Proposition 3.1, we have the following estimation:*

$$T_5 \leq \eta_0 \int_S^T E\ dt + \bar{C}_{\eta_0}(C_u + C_{d,e})E(S) + \frac{\bar{C}_{\eta_0}^3}{\eta_0^2}(C_{d,e}C_u + C_{d,e}) + C_{d,e}, \tag{93}$$

where $0 < \eta_0 < 1$ *is a positive constant to be chosen later and and* $\bar{C}_{\eta_0}$ *is an implicit positive constant that depends on* η_0 *only.*

***Proof of Lemma 3.10*:** For every $R_2 > 1$, we have that

$$\begin{aligned} T_5 &\leq \frac{1}{a_0} \int_S^T \int_\omega a(x) u_t^2\, dx\, dt \leq C \int_S^T \int_\Omega a(x)(u_t + d)^2\, dx\, dt + C \int_S^T \int_\Omega a(x) d^2\, dx\, dt \\ &\leq C \underbrace{\int_S^T \int_{|u_t+d| \leq R_2} a(x)(u_t + d)^2\, dx\, dt}_{Z_1} + C \underbrace{\int_S^T \int_{|u_t+d| > R_2} a(x)(u_t + d)^2\, dx\, dt}_{Z_2} + CC_1(d). \end{aligned} \tag{94}$$

On one hand, since $g'(0) > 0$, there exists $\alpha_{R_2} > 0$ such that $|g(v)| \geq \alpha_{R_2}|v|$ for $|v| \leq R_2$. Combining that with (66) yields for some $0 < \eta_6 < 1$

$$
\begin{aligned}
Z_1 &\leq \int_S^T \int_{|u_t+d|\leq R_2} a(x)(u_t+d)g(u_t+d)\frac{(u_t+d)}{g(u_t+d)}\,dx\,dt \\
&\leq \frac{1}{\alpha_{R_2}}\int_S^T \int_{|u_t+d|\leq R_2} a(x)(u_t+d)g(u_t+d)\,dx\,dt \\
&\leq \frac{1}{\alpha_{R_2}}\int_S^T \int_\Omega a(x)(u_t+d)g(u_t+d)\,dx\,dt \\
&\leq \frac{1}{\alpha_{R_2}}E(S) + C\frac{1}{\alpha_{R_2}}\eta_6^2\int_S^T E\ dt + \frac{1}{\alpha_{R_2}}\frac{1}{\eta_6^4}(C_{d,e}C_u + C_{d,e}).
\end{aligned}
$$

We choose $\eta_6 = \sqrt{\frac{\alpha_{R_2}}{C}\eta_0}$, we obtain that

$$
Z_1 \leq \frac{1}{\alpha_{R_2}}E(S) + \eta_0\int_S^T E\ dt + \frac{1}{\alpha_{R_2}^3\eta_0^2}(C_{d,e}C_u + C_{d,e}).
$$

As for Z_2, we have that

$$
\begin{aligned}
Z_2 &\leq C\int_S^T \int_{|u_t+d|>R_2} |u_t|^2\,dx\,dt + C\int_S^T \int_{|u_t+d|>R_2} |d|^2\,dx\,dt \\
&\leq C\int_S^T \int_{|u_t+d|>R_2} \frac{|u_t+d|}{R_2}|u_t|^2\,dx\,dt + CC_1(d) \\
&\leq C\int_S^T \int_{|u_t+d|>R_2} \frac{|u_t|^3}{R_2}\,dx\,dt + C\int_S^T \int_{|u_t+d|>R_2} \frac{|u_t|^2|d|}{R_2}\,dx\,dt + CC_1(d) \\
&\leq \frac{C}{R_2}\int_S^T \int_{|u_t+d|>R_2} |u_t|^3\,dx\,dt + \frac{C}{R_2^2}\int_S^T \int_{|u_t+d|>R_2} |u_t|^4\,dx\,dt + CC_1(d).
\end{aligned} \tag{95}
$$

We use Lemma 3.3 and the fact that $R_2 > 1$, we derive the following:

$$
\frac{C}{R_2}\int_S^T \int_{|u_t+d|>R_2} |u_t|^3\,dx\,dt + \frac{C}{R_2^2}\int_S^T \int_{|u_t+d|>R_2} |u_t|^4\,dx\,dt \leq \left(\frac{C_u + C_{d,e}}{R_2}\right)\int_S^T E\ dt. \tag{96}
$$

We choose $R_2 = \frac{(C_u + C_{d,e})}{\eta_0}$ and we combine (95) and (96) we have that

$$
Z_2 \leq \eta_0\int_S^T E\ dt + C_{d,e}. \tag{97}
$$

Remark 3.5 *For such a choice of R_2, and based on how α_{R_2} is defined, we can assume that $\frac{1}{\alpha_{R_2}}$ is also a constant of the type $\bar{C}_{\eta_0}(C_u + C_{d,e})$, where $\bar{C}_{\eta_0}$ is a constant that depends on η_0 only. As a result, Z_1 is estimated by*

$$
Z_1 \leq \eta_0\int_S^T E\ dt + \bar{C}_{\eta_0}(C_u + C_{d,e})E(S) + \frac{\bar{C}_{\eta_0}^3}{\eta_0^2}\left(C_{d,e}C_u + C_{d,e}\right). \tag{98}
$$

Combining (94), (97) and (98) and using (10) and (13), it follows that

$$T_5 \leq \eta_0 \int_S^T E\ dt + \bar{C}_{\eta_0}(C_u + C_{d,e})E(S) + \frac{\bar{C}_{\eta_0}^3}{\eta_0^2}(C_{d,e}C_u + C_{d,e}) + C_{d,e},$$

which proves Lemma 3.10.

■

The estimation of T_5 gives a direct estimation of the term $\frac{C}{\eta_0}\int_S^T \int_\omega u_t^2\, dx\, dt$ left in the estimation of T_2. We can easily manage to have that

$$\frac{1}{\eta_0}\int_S^T \int_\omega u_t^2\, dx\, dt \leq \eta_0 \int_S^T E\ dt + \frac{1}{\eta_0}\bar{C}_{\eta_0^2}(C_u + C_{d,e})E(S) + \frac{1}{\eta_0}\frac{\bar{C}_{\eta_0^2}^3}{\eta_0^4}(C_{d,e}C_u + C_{d,e}) + C_{d,e}. \quad (99)$$

It is obtained by following the same steps that led to the estimation of T_5 with replacing η_0 by η_0^2.

We can finally finish the proof of Proposition 3.1: we combine the estimations of $T_i,\ i = 1, 2, 3, 4, 5$, which are given by (42), (43), (81), (89) and (93) with (41), then we choose η_0 such that $C\eta_0 < 1$, which means that the term $C\eta_0 \int_S^T E(t)\, dt$ gets absorbed by $\int_S^T E(t)\, dt$. Then we use the fact that $C_{d,e}E(S) \leq C_{d,e}(E(0) + C_{d,e}) = C_{d,e}C_u + C_{d,e}$ and the fact that the choice of η_0 will be a constant C, we obtain (39).

■

Proof of the energy estimate of Theorem 2.1: Using the key result given by (39), we get at once from Theorem A.1 that (101) holds true with $T = C + C_u$ and $C_0 = (1 + C_u)C_{d,e}$. Using (20) for $t \geq 1$ with $T = t$ and $S \in [t-1, t]$ and integrating it over $[t-1, t]$, one gets that

$$E(t) \leq C \int_{t-1}^t E(s)\ ds + C_{d,e} \leq C \int_{t-1}^\infty E(s)\ ds + C_{d,e}.$$

Combining the above with (100) yields (18) for $t \geq 1$. In turn, (20) with $T \in [0, 1]$ and $S = 0$ provides (18) for $t \leq 1$. The proof of Theorem 2.1 is then completed.

■

A Appendix

We list in what follows, technical results used in the core of the paper.

Theorem A.1 ***Gronwall integral lemma***
Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy for some $C_0, T > 0$:

$$\int_t^{+\infty} E(s)ds \leq TE(t) + C_0, \quad \forall\ t \geq 0. \quad (100)$$

Then, the following estimate hold true

$$\int_t^{+\infty} E(s)ds \leq TE(0)e^{-\frac{t}{T}} + C_0, \quad \forall\ t \geq 0. \quad (101)$$

If in addition, $t \mapsto E(t)$ is non-increasing, one has

$$E(t) \leq E(0)e^{1-\frac{t}{T}} + \frac{C_0}{T}, \quad \forall\ t \geq 0. \quad (102)$$

The proof is classical, cf. for instance [3].

Theorem A.2 ***Generalized Gronwall lemma***
Let F, h_1 and h_2 non negative functions defined on $\mathbb{R}_+$ satisfying

$$\|h_1\|_1 := \int_0^\infty h_1(t)dt < \infty, \quad \|h_2\|_1 := \int_0^\infty h_2(t)dt < \infty,$$

and

$$F(T) \leq F(S) + C_3 + C_1 \int_S^T h_1(s)F^{\alpha_1}(s)ds + C_2 \int_S^T h_2(s)F^{\alpha_2}(s)ds, \quad \forall\ S \leq T, \tag{103}$$

where C_1, C_2, C_3 are positive constants and $0 \leq \alpha_1, \alpha_2 < 1$. Then, F satisfies the following bound

$$\sup_{t\in[S,T]} F(t) \leq \max\left(2(F(S) + C_3), (2\tilde{C})^{\frac{1}{1-\alpha}}\right), \text{ with } \tilde{C} := C_1\|h_1\|_1 + C_2\|h_2\|_1, \tag{104}$$

where $\alpha := \max(\alpha_1, \alpha_2)$ if $2\tilde{C} \geq 1$ or $\alpha := \min(\alpha_1, \alpha_2)$ if $2\tilde{C} < 1$.

Proof of Theorem A.2: Fix $T \geq S \geq 0$. For $t \in [S,T]$ set $Y(t)$ for the right-hand side of (103) applied at the pair of times $S \leq t$. It defines a non decreasing absolutely continuous function. Since $F(t) \leq Y(t) \leq Y(T)$ for $t \in [S,T]$, one deduces that $F_{S,T} := \sup_{t\in[S,T]} F(t)$ is finite for every $t \in [S,T]$. One gets from (103) that

$$F_{S,T} \leq F(S) + C_3 + \tilde{C}\max(F_{S,T}^{\alpha_1}, F_{S,T}^{\alpha_2}),$$

with the notations of (104). The latter follows at once by considering whether $F(S) + C_3 > \tilde{C}\max(F_{S,T}^{\alpha_1}, F_{S,T}^{\alpha_2})$ or not.

We recall the following useful result, cf. for instance [16].

Theorem A.3 ***Gagliardo–Nirenberg interpolation inequality***
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $N \geq 1$, $1 \leq r < p \leq \infty, 1 \leq q \leq p$ and $m \geq 0$. Then the inequality

$$\|v\|_p \leq C\|v\|_{m,q}^{\theta}\|v\|_r^{1-\theta} \qquad for \quad v \in W^{m,q}(\Omega) \cap L^r(\Omega) \tag{105}$$

holds for some constant $C > 0$ and

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1}, \tag{106}$$

where $0 < \theta \leq 1$ $(0 < \theta < 1$ if $p = \infty$ and $mq = N)$ and $\|.\|_p$ denotes the usual $L^p(\Omega)$ norm and $\|.\|_{m,q}$ the norm in $W^{m,q}(\Omega)$.

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