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Clarisse Alboin, Jérôme Jaffré, Patrick Joly, Jean Roberts

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On a convolution operator arising in a double porosity model

Clarisse Alboin* , Jérôme Jaffré†‡ , Patrick Joly†§ , Jean Roberts†¶

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Abstract: Contaminant transport in a fractured porous medium can be modeled, under appropriate conditions, with a double porosity model. Such a model consists of a parabolic equation with a coupling term describing contaminant exchange between the fractures, which have high permeability, and the matrix block, which has low permeability. Interpreting the coupling term as an operator, pseudo-differential in time, we obtain a method for calculating this term which is rapid in comparison with standard discretization methods.

Key-words: Porous medium, fractures, coupling operator, convolution operator

* Cea-Saclay, DMT/SEMT/MTMS, 91191 Gif sur Yvette Cedex, France
† Inria-Rocquencourt, BP 105, 78153 Le Chesnay Cedex, France
‡ Jerome.Jaffre@inria.fr
§ Patrick.Joly@inria.fr
¶ Jean.Roberts@inria.fr

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 —Télécopie : +33 1 39 63 53 30

Un opérateur de convolution provenant d'un modèle double porosité

Résumé : Le transport de contaminants dans le sous-sol peut, dans certaines conditions, être modélisé par un modèle double porosité. Un tel modèle consiste en une équation parabolique avec un terme de couplage décrivant l'échange de contaminants entre les fractures, qui ont une perméabilité élevée, et les blocs matriciels, dont la perméabilité est au contraire très faible. En interprétant ce terme de couplage comme un opérateur pseudo-différentiel en temps, nous obtenons une méthode pour le calculer qui est très efficace en comparaison de méthodes de discrétisation standards.

Mots-clés : Milieu poreux, fractures, couplage, opérateur de convolution

1 Physical motivation

We consider the problem of simulating the transport of a contaminant through a naturally fractured porous medium. The fractures are assumed to be connected and too numerous to be modeled individually. The source term is located in the network of fractures and vanishes after a certain time. Thus, in a first stage, the contaminant, in addition to being convected and diffused in the network of the fractures, is dispersed by diffusion into the blocks of porous media, called matrix blocks, which act as reservoirs. Then in a second stage, after the concentration of contaminant in the fractures has diminished, the matrix block acts as a source as the contaminant is rediffused into the fracture system. These exchanges between matrix blocks and fractures are very significant and modify the propagation time of the contaminant [1]. The difficulty is to evaluate the coupling term which governs the matrix block/fracture transfers.

2 The double porosity model

Three scales exist naturally in the medium, the scale of the thickness of the fracture, the scale of the average distance between fracture planes, and the scale of the medium. The latter scale is very large, so tractable computer simulation requires that effects on the two finer scales be averaged. The difficulty is in averaging the system in such a way that the transfer of contaminant is taken into account correctly. In several studies [2, 3, 4], an averaging medium is derived from the initial medium by homogenization. We consider a naturally fractured medium throughout which exists a system of interconnected fracture planes. This medium is idealized as a periodic medium as shown in Fig. 1.

We carry out the homogenization for this idealized medium to obtain a medium which has the average properties of the initial medium. We say that the scale of heterogeneities, is the microscopic scale and the scale of the homogenized medium is the macroscopic scale. The resulting model is composed of a concentration equation in a homogenized medium to which has been added a source term or coupling term whose value at point x and time t is obtained by solving a diffusion equation in a matrix block. The solution acts itself as a source term for the microscopic model in the matrix block. This model is the double porosity model. Two sets of parameters describe the homogenized medium: macroscopic parameters derived from the microscopic parameters in the fractures and microscopic parameters in the matrix blocks from which the coupling term is determined.

The double porosity model was first described in articles [5, 6], where the model was a phenomenological model deduced from experimentation. The model was later derived via homogenization.

We denote by x the macroscopic variable and by y the microscopic variable (Fig. 2).

The calculation of the coupling term involves, at each point $x \in \Omega$, the solution of an equation modeling diffusion in a matrix block. We denote by Ω the domain, by $\mathcal{Q}_x^m$ the matrix block associated with x, by $\mathcal{Q}_x$ a matrix block surrounded by a fracture (Fig. 3) and by T the length of time of the simulation.

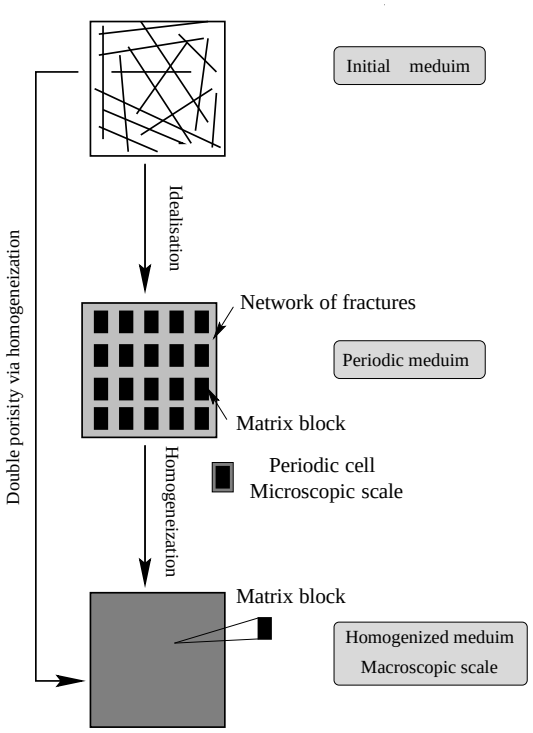

Figure 1: The double porosity model

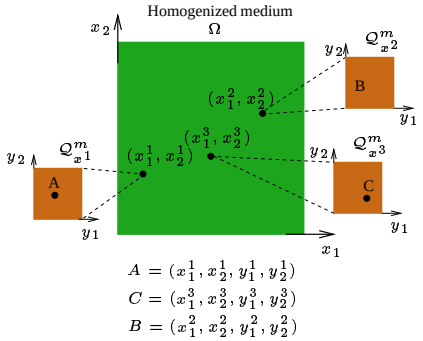

Figure 2: Systems of variables

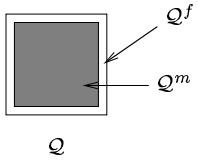

Figure 3: The fracture Q^f surrounding a matrix bloc Q^m and $Q = Q^m \cup Q^f$

After homogenization, we obtain the following model, for $x \in \Omega$,

$$\begin{aligned} \frac{\partial c(x,t)}{\partial t} + \mathrm{div}_x[-D(x)\nabla_x c(x,t) + u(x)c(x,t)] \\ = q_c(x,t) - \frac{1}{|\mathcal{Q}_x|}\int_{\mathcal{Q}_x^m} \frac{\partial c^m(x,y,t)}{\partial t}\, dy \\ c(x,0) = c^0(x) \end{aligned} \tag{1}$$

where the unknown $c(x,t)$ is the concentration in the homogenized medium and D, u and q_c are respectively the diffusion coefficient, the Darcy velocity and the given source term in the homogenized medium. The term

$$\frac{1}{|\mathcal{Q}_x|}\int_{\mathcal{Q}_x^m} \frac{\partial c^m(x,y,t)}{\partial t}\, dy,$$

is the coupling term which represents the exchange between the fractures and the matrix blocks. The unknown $c^m(x,y,t)$ is the concentration at time t and at point y of the matrix block Q_x associated

with x in Ω and is the solution of the contaminant transport equation on the microscopic scale: for $x \in \Omega$, $y \in \mathcal{Q}_x^m$,

$$\begin{aligned} &\frac{\partial c^m}{\partial t}(x,y,t) - \alpha(x)\Delta_y c^m(x,y,t) \;=\; 0 \\ &c^m(x,y,0) \;=\; 0 \\ &c^m(x,y,t) \;=\; c(x,t) \quad \text{ on } \partial Q_x^m, \end{aligned} \tag{2}$$

where α is the diffusion coefficient in the matrix block. Here the advection term has been neglected as the permeability is very low in Q_x^m. In the numerical model, (2) must be solved at each point x of the discretized domain Ω and at each time step. Solving (2) by a numerical method is costly as the low diffusion in the matrix block renders the problem quite stiff and necessitates the use of a fine grid. This led us to calculate the coupling term analytically. The idea is to reformulate the coupling term as a function of the concentration in the homogenized medium.

3 The coupling operator as an operator pseudo-differential in time

For each $\alpha \in \mathbb{R}^+$, we introduce the linear operator, $\tilde{\mathcal{S}}_\alpha$,

$$\begin{aligned} \tilde{\mathcal{S}}_\alpha \;:\; H^1(0,T) &\longrightarrow L^2(0,T) \\ \varphi &\longrightarrow \tilde{\mathcal{S}}_\alpha(\varphi), \end{aligned}$$

where

$$S_\alpha(\varphi)(t) \;=\; \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}^m} \frac{\partial \varphi^m}{\partial t}(y,t)\,dy, \tag{3}$$

and where $\varphi^m \in H^1(0,T;L^2(Q^m)) \cap L^2(0,T;H^1(Q^m))$ is the solution of the following problem:

$$\begin{aligned} &\frac{\partial \varphi^m}{\partial t}(y,t) - \alpha\,\Delta_y \varphi^m(y,t) \;=\; 0 \quad \text{in } Q^m \\ &\varphi^m(y,0) = 0 \qquad\qquad\qquad\qquad \text{in } Q^m \\ &\varphi^m(y,t) = \varphi(t) \qquad\qquad\qquad\quad \text{on } \partial Q^m. \end{aligned} \tag{4}$$

Then for α a function from Ω to $\mathbb{R}^+$, we denote by $\mathcal{S}_\alpha$ the operator

$$\begin{aligned} \mathcal{S}_\alpha \;:\; H^1(0,T;L^2(\Omega)) &\longrightarrow L^2(0,T;L^2(\Omega)) \\ c &\longrightarrow \mathcal{S}_\alpha(c) \end{aligned} \tag{5}$$

where for almost every $x \in \Omega$,

$$\mathcal{S}_\alpha(c)(x,.) = \tilde{\mathcal{S}}_{\alpha(x)}(c(x,\cdot)). \tag{6}$$

With this notation, we can rewrite problem (1) as follows:

$$\begin{aligned} &\frac{\partial c}{\partial t} + \mathrm{div}_x[-D(x)\nabla_x c + u(x)c] + \mathcal{S}_\alpha(c) \;=\; q_c \\ &c(x,0) \;=\; c^0(x). \end{aligned} \tag{7}$$

As, in (6), the coefficient is constant in each matrix block and independent of time, the Dirichlet condition is constant on the boundary of the matrix block and the initial concentration is constant on the matrix block, the coupling term $\tilde{\mathcal{S}}_\alpha(\varphi)$ may be expressed as a convolution in time: the analytic solution of (4) is calculated and then substituted into (3) yielding

$$\begin{aligned} \tilde{\mathcal{S}}_\alpha(\varphi)(t) \;=\; \frac{1}{|\mathcal{Q}|}[\int_0^t \frac{d\varphi(s)}{dt} K_\alpha(t-s)\,ds \\ + K_\alpha(t)\varphi(0)], \end{aligned} \tag{8}$$

where K_α is the convolution kernel,

$$\begin{aligned} K_\alpha(t) \;=\; \alpha\pi^2 \sum_{p,q\geq 1} (\mu_p\mu_q)^2 \frac{(p^2+q^2)}{|\mathcal{Q}^m|} \\ exp\left(-\frac{\alpha\pi^2}{|\mathcal{Q}^m|}(p^2+q^2)t\right) \end{aligned} \tag{9}$$

with

$$\mu_{2p} \;=\; 0 \qquad \mu_{2p+1} \;=\; (-1)^p \frac{2\sqrt{2}|\mathcal{Q}^m|^{\frac{1}{4}}}{(2p+1)\pi}.$$

Expressions (8) and (9) are obtained by rewriting φ^m as

$$\varphi^m(y,t) = \varphi(t) + \psi^m(y,t),$$

where ψ^m is the solution of

$$\begin{aligned} &\frac{\partial \psi^m}{\partial t}(y,t) - \alpha\,\Delta_y\psi^m(y,t) \;=\; -\frac{d\varphi}{dt}(t) && \text{in } Q^m \\ &\psi^m(y,0) \;=\; -\varphi(0) && \text{in } Q^m \\ &\psi^m(y,t) \;=\; 0 && \text{on } \partial Q^m, \end{aligned} \tag{10}$$

and expanding ψ^m in terms of a basis of eigenfunctions of the Laplace operator in Q^m with homogeneous boundary conditions.

We have the following properties:

(i) The series, K_α converges on $[0,+\infty[$ and converges uniformly on the subinterval $[\varepsilon,+\infty[$ for each $\varepsilon > 0$. Moreover K_α has the following behavior, for $\alpha \in \mathbb{R}^+$ given,

$$K_\alpha(t) = 4\sqrt{\frac{|\mathcal{Q}^m|\alpha}{\pi t}} - \frac{16\alpha}{\pi} + O(t). \tag{11}$$

(ii) As a consequence of (11), $\mathcal{S}_\alpha$ is a pseudo-differential time operator of order $1/2$, in the sense that, for all $s < \frac{1}{2}$,

$$\mathcal{S}_\alpha \ : \ H^s(\mathbb{R}^+; L^2(\Omega)) \longrightarrow H^{s-\frac{1}{2}}(\mathbb{R}^+; L^2(\Omega)).$$

(iii) As t increases away from 0, the kernel K_α decreases rapidly,

$$K_\alpha(t) \leq C\, e^{\frac{-2\alpha\pi^2 t}{|\mathcal{Q}^m|}}. \tag{12}$$

(iv) The operator, $\mathcal{S}_\alpha$, is semi-positive in the sense that for almost every $x \in \Omega$

$$\int_0^T \mathcal{S}_\alpha(c)(x,s)c(x,s)ds \geq 0. \tag{13}$$

Another fundamental property is

$$\int_0^T \mathcal{S}_\alpha(c)(x,s)\frac{\partial c(x,s)}{\partial s}ds \geq 0. \tag{14}$$

Either of (13) and (14) suffices to show that problem (7) is well posed.

We give the main ideas of the proof; for more details see [7]. For simplicity of notation we introduce $h = \sqrt{|\mathcal{Q}^m|}$.

(i) The behavior (11) is obtained by exploiting another expression for the kernel $K_\alpha(t)$. Note that

$$\begin{aligned} K_\alpha(t) \ &= \ -2\zeta(t)\ \zeta'(t) \\ \text{where} \quad \zeta(t) \ &= \ \sum_{p\geq 1} \mu_p^2\, e^{-\frac{\alpha(x)\pi^2}{h^2}p^2\, t}. \end{aligned} \tag{15}$$

Now, $\zeta(t)$ is also given by

$$\zeta(t) \ = \ \int_{-\frac{h}{2}}^{\frac{h}{2}} z(y,t)\, dy,$$

where $z(y,t)$ is the solution of

$$\begin{aligned} &\frac{\partial z}{\partial t} - \alpha \frac{\partial^2 z}{\partial y^2} = 0 \quad -h/2 \le y \le h/2 \\ &z(y,0) = -1 \qquad -h/2 \le y \le h/2 \\ &z(\pm h/2, t) = 0. \end{aligned} \tag{16}$$

By the image principle, $z(y,t)$ can be identified with the restriction to the segment $[-h/2, h/2]$ of the solution $\tilde{z}(y,t)$ of

$$\begin{aligned} &\frac{\partial \tilde{z}}{\partial t} - \alpha \frac{\partial^2 \tilde{z}}{\partial y^2} = 0 \quad y \in \mathbb{R},\ t > 0 \\ &\tilde{z}(y,0) = \sum_l (-1)^{l+1}\, \chi(y - lh) \quad y \in \mathbb{R}. \end{aligned} \tag{17}$$

Therefore

$$\tilde{z}(y,t) = \sum_{l=-\infty}^{l=+\infty} (-1)^{l+1} z_l(y,t) \tag{18}$$

with

$$z_l(y,t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-h/2+hl}^{h/2+hl} e^{-\frac{(y-\eta)^2}{4t\alpha}} d\eta. \tag{19}$$

(ii) Property (ii) can be proved by using the Fourier transform of the coupling term. We denote the Fourier transform by $\mathcal{F}$ and let $\tilde{c} \in L^2(\mathbb{R}, L^2(\Omega))$, be the extension to $\mathbb{R}$ of c defined by $\tilde{c} = c$ on $[0, +\infty[$ and $\tilde{c} = 0$ on $]-\infty, 0[$. Using relations (3) and (6) and writing the Fourier transform of $\tilde{c}$ in terms of a basis of eigenfunctions for the Laplace operator, one sees that for almost all $x \in \Omega$,

$$\begin{aligned} \mathcal{F}(\mathcal{S}_\alpha(\tilde{c}))(x,\omega) = -\frac{i\,\omega}{|\mathcal{Q}|} \Bigg[&\sum_{p\ge 1, q\ge 1} \mu_{pq}^2 \\ &\frac{\alpha\,\pi^2(p^2+q^2)}{-i\,\omega|\mathcal{Q}^m| + \alpha\,\pi^2(p^2+q^2)} \Bigg]\ \mathcal{F}(\tilde{c})(x,\omega). \end{aligned}$$

Defining $\mathcal{F}(\mathcal{S}_\alpha)$ by

$$\begin{aligned} \mathcal{F}(\mathcal{S}_\alpha)(\omega) = -\frac{i\,\omega}{|\mathcal{Q}|} \Bigg[&\sum_{p\ge 1, q\ge 1} \mu_{pq}^2 \\ &\frac{\alpha\,\pi^2(p^2+q^2)}{-i\,\omega|\mathcal{Q}^m| + \alpha\,\pi^2(p^2+q^2)} \Bigg] \end{aligned} \tag{20}$$

one can write, for almost all $x \in \Omega$,

$$\mathcal{F}(\mathcal{S}_\alpha(\tilde{c}))(x,\omega) \;=\; \mathcal{F}(\mathcal{S}_\alpha)(\omega)\,\mathcal{F}(\tilde{c})(x,\omega). \tag{21}$$

The series in (20) converges uniformly in ω on each compact interval in $\mathbb{R}$ so $\mathcal{F}(\mathcal{S}_\alpha)$ is continuous on each compact interval in $\mathbb{R}$. We note that

$$\lim_{\omega\to+\infty} \mathcal{F}(\mathcal{S}_\alpha)(\omega) \;=\; +\infty.$$

The behavior of $\mathcal{F}(\mathcal{S}_\alpha)$ when ω increases toward infinity, may be determined by using (8) and (6) to write the Fourier transform of the coupling term in terms of the kernel K_α :

$$\mathcal{F}(\mathcal{S}_\alpha(\tilde{c}))(x,\omega) \;=\; -\frac{i\omega}{|\mathcal{Q}|}\mathcal{F}(\tilde{c})(x,\omega)\mathcal{F}(K_\alpha)(\omega).$$

It follows from (21) that

$$\mathcal{F}(\mathcal{S}_\alpha)(\omega) \;=\; -\frac{i\omega}{|\mathcal{Q}|}\mathcal{F}(K_\alpha)(\omega). \tag{22}$$

It can now be shown that

$$\mathcal{F}(K_\alpha)(\omega) \sim_{(\omega\to+\infty)} 4\sqrt{\frac{|\mathcal{Q}^m|\alpha}{\pi}}\frac{\hat{\rho}}{\sqrt{\omega}},$$

with $\hat{\rho} = \displaystyle\int_0^{+\infty} \frac{e^{-iu}}{\sqrt{u}}\,du$, and it follows from (22) that

$$\mathcal{F}(\mathcal{S}_\alpha)(\omega) \sim_{(\omega\to+\infty)} -4i\sqrt{\frac{|\mathcal{Q}^m|\alpha}{\pi}}\frac{\hat{\rho}}{|\mathcal{Q}|}\sqrt{\omega}. \tag{23}$$

Finally, as the Fourier transform of the coupling term is continuous in ω on each compact interval in $\mathbb{R}$ and behaves as does $\sqrt{\omega}$ as ω increases toward infinity, there exists a constant such that

$$|\mathcal{F}(\mathcal{S}_\alpha(\varphi))|(\omega) \le cste(1+\omega^2)^{\frac{1}{4}}. \tag{24}$$

By definition,

$$||c||^2_{H^s(\mathbb{R}^+;L^2(\Omega))} \;=\; \int_{\mathbb{R}} (1+\omega^2)^s |\mathcal{F}(\tilde{c})(x,\omega)|^2 d\omega,$$

and from (24), it follows that

$$||c||^2_{H^s(\mathbb{R}^+;L^2(\Omega))} \ge$$

$$cste \int_{\mathbb{R}} (1+\omega^2)^{s-\frac{1}{2}} |\mathcal{F}(\mathcal{S}_\alpha)(\omega)|^2 |\mathcal{F}(\tilde{c})(x,\omega)|^2 d\omega.$$

We deduce from (21) that for almost all $x \in \Omega$

$$||c||^2_{H^s(\mathbb{R}^+;L^2(\Omega))} \geq cste\, ||\mathcal{S}_\alpha(\tilde{c})||^2_{H^{s-\frac{1}{2}}(\mathbb{R};L^2(\Omega))}.$$

To conclude, we prove that

$$||c||^2_{H^s(\mathbb{R}^+;L^2(\Omega))} \geq cste\, ||\mathcal{S}_\alpha(c)||^2_{H^{s-\frac{1}{2}}(\mathbb{R};L^2(\Omega))}.$$

(iii) Property (iii) is an immediate consequence of the definition of the kernel (9).

(iv) To obtain inequality (13), we multiply the first equation of (2) by c^m (solution of the system (2)) and we integrate the equation on $\mathcal{Q}^m \times [0,T]$. We obtain, by definition of $\mathcal{S}_\alpha$, the equality

$$\alpha|\mathcal{Q}| \int_0^T \mathcal{S}_\alpha(c)(x,s)\, c(x,s)\, ds \;=\; \frac{1}{2}\, ||c^m(x,y,t)||^2 + \alpha \int_0^T ||\nabla_y c^m(x,y,t)||^2\, dt.$$

To obtain inequality (14), we multiply the first equation of system (2) by $\dfrac{\partial c^m}{\partial t}$ and we integrate the equation on $\mathcal{Q}^m \times [0,T]$,

$$\alpha|\mathcal{Q}| \int_0^T \mathcal{S}_\alpha(c)(x,s) \frac{\partial c^m(x,y,s)}{\partial s} ds \;= \int_0^T ||\frac{\partial c(x,s)}{\partial s}||^2 ds + \frac{\alpha}{2}||\nabla_y c^m(x,y,t)||^2.$$

In practice, expression (11) for K_α is very useful for evaluating K_α in a neighborhood of the singularity at 0 (the series in (9) converges very slowly for small t).

Property (iii) will be exploited for the approximation of the coupling term (see section 4).

4 Numerical approximation of the coupling operator

4.1 Time discretization

Denoting by Δt the length of the time step in the homogenized medium we write the semi-discretization in time of (7) as follows

$$\frac{c^{n+1} - c^n}{\Delta t} + \mathrm{div}(-D\nabla c^{n+1} + u\, c^n) + (S_\alpha c)^{n+\frac{1}{2}} \;=\; q_c^{n+1}, \tag{25}$$

where the approximation $(S_\alpha c)^{n+\frac{1}{2}}$ of the coupling term, at a point $x \in \Omega$, is given by

$$(S_\alpha c)^{n+\frac{1}{2}}(x) = [\sum_{k=0}^{n} K_{nk}(c^{k+1}(x) - c^k(x)) + K_n c^0(x)] \frac{1}{|\mathcal{Q}|}, \tag{26}$$

and where

$$\begin{aligned} K_{nk} &= \int_{n\Delta t}^{(n+1)\Delta t} \int_{k\Delta t}^{(k+1)\Delta t} \frac{K_\alpha(t-s)}{(\Delta t)^2}\, ds\, dt \\ &\quad \forall n \text{ and } k = 0, ..., n-1, \\ K_{nn} &= \int_{n\Delta t}^{(n+1)\Delta t} \int_{n\Delta t}^{t} \frac{K_\alpha(t-s)}{(\Delta t)^2}\, ds\, dt \ \ \forall n, \\ K_n &= \int_{n\Delta t}^{(n+1)\Delta t} \frac{K_\alpha(t)}{\Delta t}\, dt \ \ \forall n. \end{aligned}$$

This discretization in time is chosen in order to guarantee the stability of the scheme. In fact, the discretization in time of the coupling term satisfies the stability condition

$$\sum_{n=0}^{Nt-1} (\mathcal{S}_\alpha c)^{n+\frac{1}{2}} \frac{c^{n+1}(x) - c^n(x)}{\Delta t} \Delta t \geq 0, \tag{27}$$

where $Nt = \frac{T}{\Delta t}$ is the number of time steps. To see this inequality, set

$$c_{\Delta t}(x,t) = \sum_{l=0}^{Nt} c^l(x)\omega_l(t), \tag{28}$$

where ω_l is a hat function,

$$\begin{cases} \omega_0 &= -\dfrac{t}{\Delta t} + 1 \quad \text{for} \quad t = [0, \Delta t] \\ \omega_{NT} &= \dfrac{t}{\Delta t} + (1 - Nt) \quad \text{for} \quad t = [(Nt-1)\Delta t, Nt\Delta t] \end{cases}$$

and for $1 \leq l \leq Nt - 1$,

$$\begin{cases} \omega_l &= \dfrac{t}{\Delta t} + (1 - l) \quad \text{pour} \quad t = [(l-1)\Delta t, l\Delta t] \\ \omega_l &= -\dfrac{t}{\Delta t} + (1 + l) \quad \text{pour} \quad t = [l\Delta t, (l+1)\Delta t]. \end{cases}$$

We note that

$$\frac{\partial\, c_{\Delta t}(x,t)}{\partial t} \;=\; \sum_{l=0}^{Nt-1} \frac{c^{l+1}(x)-c^{l}(x)}{\Delta t}\zeta_{l+1/2}(t), \tag{29}$$

where $\zeta_{l+1/2}(t)$ is the characteristic function

$$\zeta_{l+1/2}(t) \;=\; \begin{cases} 1 & \text{on } [l\,dt,(l+1)\,dt] \\ 0 & \text{otherwise.} \end{cases}$$

From (14), we have

$$\int_0^T \mathcal{S}_\alpha(c_{\Delta t})(x,t)\frac{\partial c_{\Delta t}(x,t)}{\partial t}\,dt \geq 0.$$

If we substitute for $c_{\Delta t}$ and $\dfrac{\partial c_{\Delta t}}{\partial t}$ their expressions given in (28) and (29), we obtain inequality (27).

4.2 Truncation of the kernel K_α

In practice, the expression for $(S_\alpha c)^{n+\frac{1}{2}}(x)$ given in (26) is still not suitable for calculation: the expression for K_α involves an infinite sum. However, for t very large, K_α is sufficiently small (see (12)) that we can replace K_α by 0. We replace the sum $\displaystyle\sum_{k=0}^{n}$ by $\displaystyle\sum_{k=n-M}^{n}$ in (26):

$$(S_\alpha c)_M^{n+\frac{1}{2}}(x) = \Big[\sum_{k=n-M}^{n} K_{nk}(c^{k+1}(x)-c^{k}(x)) + K_n c^0(x)\Big]\,\frac{1}{|\mathcal{Q}_x|},$$

and M becomes a parameter of approximation. We also point out that though the expression in (26) for $(S_\alpha c)^{n+\frac{1}{2}}(x)$ seems to require the storage in memory of the concentration at each time step, this is in fact not the case as we have set $K_\alpha = 0$ for t large which implies $K_{nk} = 0$ for k not close to n.

We conjecture that the discretized coupling term expressed with the truncated kernel satisfies as well the stability condition

$$\sum_{n=0}^{Nt-1} (\mathcal{S}_\alpha c)_M^{n+\frac{1}{2}}\,\frac{c^{n+1}(x)-c^{n}(x)}{\Delta t}\Delta t \geq 0.$$

4.3 Numerical result

The G.E. Grisak experiment : The experiment, proposed in this article consists in injecting calcium and chloride tracers into a cylinder of quasi-regular fractured porous rock; see [8]. The fractures are

distributed along two orthogonal directions, one parallel to the axis of revolution of the cylinder. The cylinder is saturated with water and the velocity of the fluid is maintained constant and parallel with the axis of revolution of the cylinder. The tracers are injected into the network of fractures at one end of the cylinder, the relative concentration depending on time is measured at the other end of the cylinder. The microscopic parameters, characterizing the rock and the network of fractures are given in Tab.1. The period of simulation is 4 days and we have 4500 time steps. To illustrate the

Parameters	*Matrix block*	*Network frac.*
Darcy velocity ($cm\,s^{-1}$)	0	$3.4375\,10^{-2}$
Longit. dispers. (cm)	-	4
Transver. dispers. (cm)	-	0
Molecular diffus. ($cm^2\,s^{-1}$)	$5\,10^{-7}$	$5\,10^{-7}$
Porosity	0.35	-
Retardation factor	1 or 10	1
Frac. aperture (cm)	-	$8\,10^{-3}$
Space of Frac. (cm)	-	4

Table 1: Parameter values for Grisak's experiment

importance of the coupling term, we simulate the transport of tracers in two cases. In the first, the coupling term is neglected and in the second it is taken into account. In Fig. 4 is shown, at the same date for each case, the concentration on a cross-section of the homogenized medium.

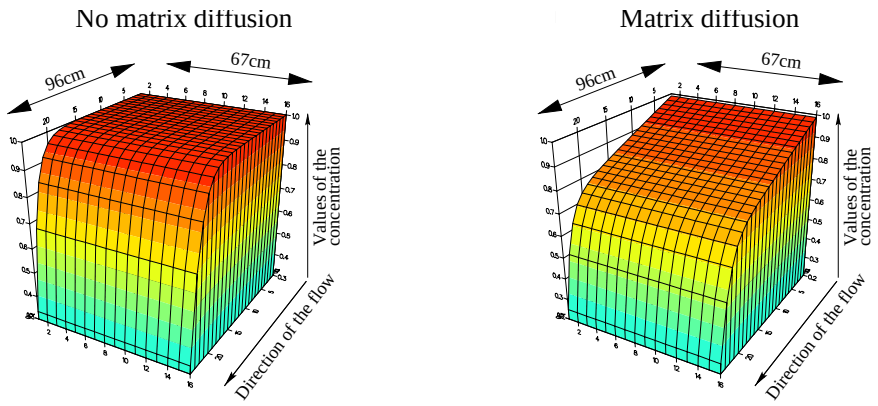

Figure 4: Concentration at 4 days, on a cross-section of the homogenized medium for data given in Table 1

We can also compare the evolution during four days of the concentration at a point of the homogenized medium, Fig. 5. One can observe that using a double porosity model (case with matrix diffusion) smears the contaminant front while with a single porosity model (case with no matrix diffusion) the front is too sharp.

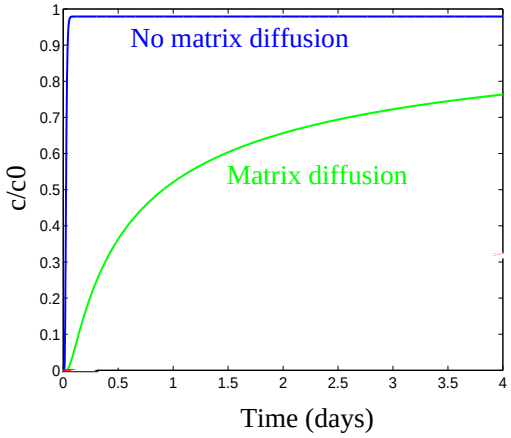

Figure 5: Relative concentration at a point of the homogenized medium for data given in Tab. 1

Comparison of different methods for calculating the coupling term:

Calculating the coupling term with a finite volume or spectral method is more costly than calculating it with the quasi-analytical method presented here. To illustrate this, we carry out a simple

experiment. We consider a square domain in which the transport is only diffusive. Initially, we have a positive concentration at the center of the domain and we let the contaminant be transported by diffusion. The data for this experiment are given in Tab. 2.

Parameters	*Matrix block*	*Network frac.*
Molecular diffus. ($cm^2\ s^{-1}$)	10^{-6}	10^{-4}
Porosity	1	-
Retardation factor	1	1
Frac. aperture (cm)	-	0.1
Space of Frac. (cm)	-	10

Table 2: Parameter values for comparison experiments

The period of simulation is $4.5\,10^7$ seconds and we use 900 time steps. We compare the run time and the memory space for three methods for calculating the coupling term. Tab 3 shows that the much shorter run time with the quasi-analytical method costs only a small increase in memory space.

Method	*Run time*	*Memory space*
Finite volume	8h26’	25 M
Spectral method	7h30’	22 M
Analytical method	2’23	28 M

Table 3: Comparison of different methods for calculating the coupling term

5 Conclusion

In a fractured porous medium with numerous interconnected fractures, the contaminant transfer between matrix blocks and fractures is very significant. This transfer is taken into account in the double porosity model with a coupling term. We have interpreted the term as an operator, pseudo-differential in time, and have calculated it with a quasi-analytical method. We have shown with numerical experiments the efficiency of this new numerical procedure.

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Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifi que
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
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