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Identification of generalized impedance boundary conditions in inverse scattering problems

Laurent Bourgeois* , Houssem Haddar†

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Abstract: In the context of scattering problems in the harmonic regime, we consider the problem of identification of some Generalized Impedance Boundary Conditions (GIBC) at the boundary of an object (which is supposed to be known) from far field measurements associated with a single incident plane wave at a fixed frequency. The GIBCs can be seen as approximate models for thin coatings, corrugated surfaces or highly absorbing media. After pointing out that uniqueness does not hold in the general case, we propose some additional assumptions for which uniqueness can be restored. We also consider the question of stability when uniqueness holds. We prove in particular Lipschitz stability when the impedance parameters belong to a compact set. We also extend local stability results to the case of back-scattering data.

Key-words: Inverse scattering problems in electromagnetism and acoustics, generalized impedance boundary conditions, stability

* Laboratoire POEMS, ENSTA, 32, Boulevard Victor, 75739 Paris Cedex 15, France
† INRIA Saclay Ile de France / CMAP Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex FRANCE

Centre de recherche INRIA Saclay – Île-de-France
Parc Orsay Université
4, rue Jacques Monod, 91893 ORSAY Cedex
Téléphone : +33 1 72 92 59 00

Identification d'impédances généralisées en problèmes de diffraction inverses

Résumé : Dans le contexte des problèmes de diffraction inverse, nous nous intéressons au problème d'identification de conditions d'impédances généralisées sur la frontière (connue) d'un objet à partir de mesures du champ lointain associé à une onde plane et pour une fréquence donnée. Les conditions d'impédances généralisées peuvent être interprétées comme des modèles approchés pour des revêtements de faible épaisseur, des surfaces rugueuses ou également pour des objets fortement absorbants. Après avoir montré, à travers un contre-exemple, que l'unicité ne peut pas être vérifiée en général, nous proposons certaines restrictions sur les paramètres de l'impédance qui permettent de restaurer l'unicité. Nous étudions également la question de la stabilité de reconstruction lorsque l'unicité a lieu. Nous montrons en particulier la stabilité lipschitzienne lorsque les paramètres appartiennent à un ensemble compact. Par ailleurs nous étendons le résultat de stabilité locale au cas de mesures en configuration de rétrodiffusion.

Mots-clés : Problèmes de diffraction inverse en électromagnétisme et en acoustique, conditions d'impédances généralisées, stabilité

1 Introduction

We address in this work uniqueness and stability issues related to the identification of a medium impedance from the knowledge of far measurements of a scattered wave at a given frequency. We restrict ourselves in these first investigations to the scalar case (the Helmholtz equation), that models either acoustic waves or two dimensional settings of electromagnetic problems. The medium impedance is understood as a "local" operator that links the Cauchy data of the field u on the medium boundary Γ. More precisely we shall consider the cases where a boundary condition of the form

$$\partial u/\partial \nu = Zu \quad \text{on} \quad \Gamma$$

is satisfied, where Z is a boundary operator and ν denotes the normal field on Γ.

The exact impedance corresponds to the so-called Dirichlet-to-Neumann (DtN) map that amounts solving the Helmholtz equation inside the medium. Determining this map is "equivalent" to determining the medium physical properties and in general this is a severely ill-posed problem that requires more than a finite number of measurements.

We are interested here in situations where the operator Z is an approximation of the exact DtN map. In general these approximations correspond to asymptotic models associated with configurations that involve a small parameter. These cases include small amplitude roughness, thin coatings, periodic gratings, highly absorbing media, ... We refer to [15, 7, 1] for a review of these models.

The simplest form is the case where Z is a scalar function, which corresponds in general to the lowest order (non trivial) approximations, for instance in the case of very rough surfaces of highly absorbing media (the Leontovich condition [12]). However, for higher order approximations or in other cases the operator Z may involve boundary differential operators. For instance when the medium contains a perfect conductor coated with a thin layer of width δ then for TM polarization, the approximate boundary conditions of order 1 corresponds to $Z = 1/\delta$ while for the TE polarization it corresponds to $Z = \delta(\partial_{ss} + k^2 n)$ where s denotes the curvilinear abscissa, k the wave number and n is the mean value of the thin coating index with respect to the normal coordinate. Higher order approximations would include curvature terms or even higher order derivatives [3, 8, 9]. This type of conditions will be referred to as Generalized Impedance Boundary Conditions.

We shall address in the present work the question of unique identification and stability of the reconstruction of the operator Z from the knowledge of one scattered wave. One easily sees, from the given example, how the identification of the impedance would provide information on some effective properties of the medium (for instance, the thickness of the coating and the normal mean value of its index). Determining these effective properties would be less demanding in terms of measurements and also more stable than solving the inverse problem with the exact DtN map. Motivated by the example above we shall consider generalized impedance boundary conditions of the form:

$$Z = \mu \Delta_\Gamma + \lambda, \tag{1}$$

where μ is a complex constant and λ is a complex function and where Δ_Γ denotes the Laplace-Beltrami operator on Γ.

The case of standard impedance problems ($\mu = 0$) has been studied by several authors [11, 16], where for instance optimal logarithmic stability is obtained. Our analysis here is different and is rather motivated by numerical considerations. It also applies to the case $\mu \neq 0$. We shall prove injectivity of the Fréchet derivative (which is equivalent to local stability) of the inverse map by using the adjoint state technique. We also investigate the situations where Lipschitz stability can be obtained. In the same spirit as in [2] we shall prove a general result showing how Lipschitz stability holds when we restrict the impedance parameters to a compact set, assuming injectivity of the inverse map and of its derivative. Motivated by this result we investigate the uniqueness question from the knowledge of one scattered wave. We show that uniqueness when $\mu \neq 0$ cannot be true in general, but holds under further assumptions on the parameters (typically assuming that part of the parameters are known) and on the geometry. Answering these questions in the case of back-scattering data would be desirable and is an open problem. We only give in that perspective a result on the injectivity of the Fréchet derivative of the inverse map.

The outline of our article is the following. We introduce in the next section the forward scattering problem and briefly recall the mathematical property of the solution in the case of GIBC. We then formulate the inverse problem. Section 3 is dedicated to the question of uniqueness from the knowledge of the far field associated to one incident wave. Section 4 is dedicated to the question of stability by analyzing the injectivity of the Fréchet derivative. It also includes a general result on Lipschitz stability. We end this work by a small excursion in the back-scattering inverse problem by extending the local stability results of Section 4 to this case.

2 The forward and inverse problems

Let D be an open bounded domain in $\mathbb{R}^3$, and $\Omega := \mathbb{R}^3 \setminus \overline{D}$. The domain D will be referred to as the obstacle and a GIBC holds on its boundary Γ. An incident plane wave $u^i(x) := e^{ik\,d.x}$, with $k > 0$ denoting the wavenumber and $||d|| = 1$, is scattered by the obstacle D and gives raise to a scattered field u^s. The governing equations for u^s in Ω are

$$\begin{cases} \Delta u^s + k^2 u^s = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u^s}{\partial \nu} + \mu \Delta_\Gamma u^s + \lambda u^s = f & \text{on} \quad \Gamma, \\ \lim\limits_{R \to +\infty} \displaystyle\int_{\partial B_R} \left| \partial u^s / \partial r - iku^s \right|^2 ds(x) = 0, \end{cases} \tag{2}$$

with

$$f := -\left(\frac{\partial u^i}{\partial \nu} + \mu \Delta_\Gamma u^i + \lambda u^i \right) |_\Gamma,$$

and where ν denotes the unit normal to Γ oriented to the exterior of D. If we set $u := u^s + u^i$ as being the total field, then the second equation of (2) is equivalent to

$$\partial u / \partial \nu + \mu \Delta_\Gamma u + \lambda u = 0 \quad \text{on} \quad \Gamma. \tag{3}$$

The last equation in (2) is the classical Sömmerfeld radiation condition where B_R denotes a ball of radius R.
We assume that the boundary Γ is Lipschitz continuous, such that for a function $v \in H^1(\Gamma)$, the tangential gradient $\nabla_\Gamma v$ is defined in $(L^2(\Gamma))^3$, and $\Delta_\Gamma v$ is defined in $H^{-1}(\Gamma)$ by the identity

$$\langle \Delta_\Gamma v, w \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} = -\int_\Gamma \nabla_\Gamma v . \nabla_\Gamma w \, ds, \quad \forall w \in H^1(\Gamma).$$

The space of solutions to the forward problem differs between the cases $\mu = 0$ and $\mu \neq 0$.

1. In the case $\mu = 0$, problem (2) coincides with the classical impedance problem. It is uniquely solvable in $V_0 := \{v \in \mathcal{D}'(\Omega),\ \phi v \in H^1(\Omega),\ \forall \phi \in \mathcal{D}(\mathbb{R}^3)\}$ provided $\lambda \in L^\infty(\Gamma)$ with $\mathrm{Im}(\lambda) \geq 0$ (see for instance [14]). This result remains true for $f \in H^{-1/2}(\Gamma)$. In [6], it is proved that if moreover the boundary Γ is C^2 and $\lambda \in C^0(\Gamma)$, then the solution u^s is continuous up to the boundary Γ.

2. In the case $\mu \neq 0$, problem (2) will be referred to as the generalized impedance problem. It is uniquely solvable in $V := \{v \in V_0,\ v|_\Gamma \in H^1(\Gamma)\}$ provided that $\lambda \in L^\infty(\Gamma)$ with $\mathrm{Im}(\lambda) \geq 0$ as well as $\mathrm{Re}(\mu) > 0$ and $\mathrm{Im}(\mu) \leq 0$. This result remains true for $f \in H^{-1}(\Gamma)$.

For sake of completeness, we shall sketch the proof in the case of the generalized impedance problem which can be seen as a slight adaptation of the proof in [14] for classical impedance problems. We restrict the problem to a bounded domain $\Omega_R := \Omega \cap B_R$ with the help of the Dirichlet-to-Neumann map $S_R : H^{1/2}(\partial B_R) \longrightarrow H^{-1/2}(\partial B_R)$, defined for $g \in H^{1/2}(B_R)$ by $S_R g = \partial_\nu u^e|_{\partial B_R}$, where u^e is the solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{B_R}$ satisfying the Sömmerfeld radiation condition and $u^e = g$ on ∂B_R. This operator satisfies in particular

$$\mathrm{Re}\,\langle S_R g, g \rangle \leq 0 \quad \text{and} \quad \mathrm{Im}\,\langle S_R g, g \rangle \geq 0 \quad \forall\, g \in H^{\frac{1}{2}}(\partial B_R),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $H^{-\frac{1}{2}}(\partial B_R)$ and $H^{\frac{1}{2}}(\partial B_R)$. Solving problem (2) is then equivalent to find u^s in $V_R := \{v \in H^1(\Omega_R);\ v|_\Gamma \in H^1(\Gamma)\}$ satisfying

$$\begin{cases} \Delta u^s + k^2 u^s = 0 & \text{in} \quad \Omega_R, \\ \dfrac{\partial u^s}{\partial \nu} + \mu \Delta_\Gamma u^s + \lambda u^s = f & \text{on} \quad \Gamma, \\ \dfrac{\partial u^s}{\partial r} = S_R(u^s|_{\partial B_R}) & \text{on} \quad \partial B_R. \end{cases} \tag{4}$$

Problem (4) is then proved to be equivalent to the weak formulation: find u^s in V_R such that for all v in V_R,

$$a(u^s, v) = l(v),$$

with

$$a(u^s, v) := \int_{\Omega_R} (\nabla u^s \cdot \nabla \overline{v} - k^2 u^s \overline{v})\, dx + \int_\Gamma (\mu \nabla_\Gamma u^s . \nabla_\Gamma \overline{v} - \lambda u^s \overline{v}\, ds - \langle S_R u^s, v \rangle\,,$$

$$l(v) = \langle f, v|_\Gamma \rangle_{H^{-1}(\Gamma),H^1(\Gamma)}\,.$$

It is easily seen that V_R equipped with the scalar product $(\cdot,\cdot)_{V_R} := (\cdot,\cdot)_{H^1(\Omega_R)} + (\cdot,\cdot)_{H^1(\Gamma)}$ is a Hilbert space. The weak formulation can then be written in the form: find u^s in V_R such that

$$(J_R + K_R)u^s = F_R \quad \text{ in } V_R$$

where the operators $J_R : V_R \longrightarrow V_R$ and $K_R : V_R \longrightarrow V_R$ are uniquely defined by

$$(J_R u^s, v)_{V_R} = (u^s, v)_{H^1(\Omega_R)} + \mu \int_\Gamma \nabla_\Gamma u^s \cdot \nabla_\Gamma \overline{v}\, ds - \langle S_R u^s, v\rangle$$

$$(K_R u^s, v)_{V_R} = -(1+k^2)(u^s, v)_{L^2(\Omega_R)} - \int_\Gamma \lambda u^s \overline{v}\, ds$$

and F_R is uniquely defined by

$$l(v) = (F_R, v)_{V_R}.$$

One then easily checks, by application of the Lax-Milgram theorem that J_R is an isomorphism for $\mathrm{Re}(\mu) > 0$, and the application of the Rellich compact embedding theorem that K_R is compact. Therefore, with the help of Fredholm alternative it is sufficient to prove the injectivity of the operator $J_R + K_R$, which is equivalent to prove uniqueness of solutions to problem (2).

Following [6], a sufficient condition for uniqueness is that for all solution $u \in V$ of (2) with $f = 0$,

$$\int_\Gamma \mathrm{Im}(\overline{u}\frac{\partial u}{\partial \nu})\, ds \leq 0.$$

Since

$$\int_\Gamma \mathrm{Im}(\overline{u}\frac{\partial u}{\partial \nu})\, ds = \mathrm{Im}(\mu)\int_\Gamma |\nabla_\Gamma u|^2\, ds - \int_\Gamma \mathrm{Im}(\lambda)|u|^2\, ds$$

we have uniqueness in the case $\mathrm{Im}(\mu) \leq 0$ and $\mathrm{Im}(\lambda) \geq 0$, which completes our sketch of proof.
Let us finally notice that both operators J_R and K_R depend continuously on λ and μ.

Formulation of the inverse problem

Following [6], the solution u^s of problem (2) has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r}(u^\infty(\hat{x}) + O(\frac{1}{r})), \quad r \to +\infty,$$

uniformly in all directions $\hat{x} = x/r \in S^2$, where $r := |x|$ and S^2 denotes the unit sphere in $\mathbb{R}^3$. The far field $u^\infty \in L^2(S^2)$ is given by

$$u^\infty(\hat{x}) = \int_\Gamma \{u^s(y)\frac{\partial \Phi^\infty(y,\hat{x})}{\partial \nu(y)} - \frac{\partial u^s}{\partial \nu}(y)\Phi^\infty(y,\hat{x})\}\, ds(y), \quad \forall \hat{x} \in S^2, \qquad (5)$$

where $\Phi^\infty(y,\hat{x}) := \frac{1}{4\pi}e^{-ik\hat{x}.y}$ and where the second integral has to be understood as a duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

The inverse problem we are interested in is to determine both μ and the function λ on Γ from the knowledge of the far field u^∞ associated with one direction d of the incident field. We shall address the questions of uniqueness and stability of this problem. For both questions we shall first consider the classical impedance problem ($\mu = 0$) and then the case of generalized impedance problem. Answering these questions amounts to study the properties of the non linear map $T : \lambda \longmapsto u^\infty$ for the first case and $T : (\lambda, \mu) \longmapsto u^\infty$ for the second one.

3 Uniqueness

This section is devoted to the investigation of the uniqueness for the inverse problem formulated above which is equivalent to analyzing the injectivity of the operator T.

3.1 The classical impedance problem

Analyzing the injectivity of T as an operator acting on $L^\infty(\Gamma)$ seems to be challenging. We shall restrict ourselves to the subspace of $L^\infty(\Gamma)$ formed by piecewise-continuous functions. Let I be a given integer and Γ_i, $i = 1, ..., I$, be open sets of Γ such that $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, $\Gamma = \cup_{i=1}^I \overline{\Gamma_i}$, and the sets $\partial\Gamma_i$ are negligible in the sense of the Lebesgue surface measure supported by Γ. We define the subspace $C_I(\Gamma)$ as the set of functions $u \in L^\infty(\Gamma)$ such that each restriction of u to Γ_i belongs to $C^0(\overline{\Gamma_i})$. The space $C_I(\Gamma)$ is obviously a closed subspace of $L^\infty(\Gamma)$.

We shall also use the subsets

$$L^\infty_+(\Gamma) := \{\lambda \in L^\infty(\Gamma)\ ;\ \mathrm{Im}(\lambda) \geq 0\} \text{ and } \mathrm{C_{I+}}(\Gamma) := \mathrm{C_I}(\Gamma) \cap \mathrm{L^\infty_+}(\Gamma).$$

Proposition 1 *The operator $T : C_{I+}(\Gamma) \to L^2(S^2)$ such that $\lambda \longmapsto u^\infty$, where u^∞ is defined from u^s by using (5) and u^s is defined from λ by solving (2), is injective.*

*Proof.*Assume that $T(\lambda_1) = T(\lambda_2) = u^\infty$ with $\lambda_1, \lambda_2 \in C_{I+}(\Gamma)$. From the Rellich Lemma and the unique continuation principle (see [6]), the function $u := u_1 - u_2 = u_1^s - u_2^s$ vanishes in Ω. Since u satisfies $\Delta u + k^2 u = 0$ in Ω, the traces $(u_1 - u_2)|_\Gamma$ and $\partial_\nu(u_1 - u_2)|_\Gamma$, which are well defined in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ respectively, also vanish.
Considering now the boundary condition on Γ,

$$\frac{\partial u_1}{\partial \nu} + \lambda_1 u_1 = \frac{\partial u_2}{\partial \nu} + \lambda_2 u_2 = 0,$$

we obtain on Γ,

$$(\lambda_1 - \lambda_2) u_1 = 0.$$

We shall prove that $\lambda_1 = \lambda_2$ almost everywhere on Γ. Assume that there exists $x_0 \in \Gamma$, where x_0 does not belong to some $\partial\Gamma_i$, such that $\lambda_1(x_0) \neq \lambda_2(x_0)$. Since λ_1 and λ_2 are continuous in a neighborhood of x_0, there exists $\eta > 0$ such that $|\lambda_1(x) - \lambda_2(x)| \geq c > 0$ for all x in $B(x_0, \eta) \cap \Gamma$. It follows that $u_1 = 0$ on $B(x_0, \eta) \cap \Gamma$. The boundary condition on Γ leads to $\partial_\nu u_1 = 0$ on $B(x_0, \eta) \cap \Gamma$,

and unique continuation leads to $u_1 = 0$ in Ω. Since the incident wave u^i does not satisfy the radiation condition, while the scattered field u_1^s does, we obtain a contradiction. □

Remark 1 *Let us notice that the proof remains valid even if I and the partition Γ_i, $i = 1, \ldots, I$ are not a priori known.*

3.2 The generalized impedance problem

We start this section by providing a counterexample showing that T cannot be injective under the sole assumptions that guarantee well-posedness of problem (2), i.e.

$$\lambda \in L^\infty(\Gamma),\ \mathrm{Im}(\lambda) \geq 0,\ \mathrm{Re}(\mu) > 0 \text{ and } \mathrm{Im}(\mu) \leq 0. \tag{6}$$

The requirements of (6) will be referred to as assumption $(H0)$ in the following. Our counterexample is valid even if the boundary Γ is C^∞ and λ is a C^∞ function on Γ.
It is constructed as follows: one first chooses a constant impedance $\lambda_0 \in i\mathbb{R}$ with $\mathrm{Im}(\lambda_0) > 0$ such that the corresponding solution u_0 of the classical impedance problem, which is C^∞ up to the boundary Γ, does not vanish on Γ. This choice is possible as demonstrated in $2D$ by the case $D = B(0,1)$, $d = (1,0)$, $k = 1$, $\lambda_0 = i$ where the solution $u_0(1,\theta)$ on Γ is given (after straightforward calculations) by

$$u_0(1,\theta) = \frac{2i}{\pi} \sum_{n \in \mathbb{Z}} \frac{i^n\, e^{in\theta}}{H_n'(1) + iH_n(1)},$$

where H_n denotes the Hankel function of the first kind of order n. The modulus of the function $u_0(1,.)$ is depicted in Figure 1. Let us denote by u_0^∞ the far field

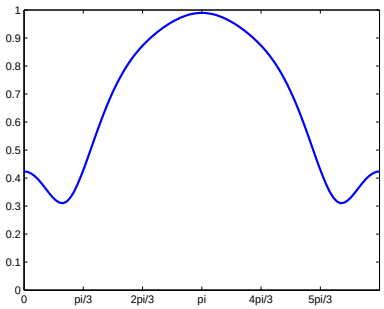


Figure 1: Modulus of $u_0(1,\theta)$ versus $\theta \in [0, 2\pi]$

associated with u_0. Since u_0 does not vanish on Γ, the function $\alpha = \Delta_\Gamma u_0 / u_0$ is a C^∞ function on Γ, and therefore bounded on Γ. Now let $\mu_1 \neq \mu_2$ be any complex numbers such that

$$|\mu_i| \max_\Gamma |\alpha| \leq \mathrm{Im}(\lambda_0), \quad \mathrm{Re}(\mu_i) > 0,\ \mathrm{Im}(\mu_i) \leq 0, \quad i = 1, 2,$$

and let the associated C^∞ functions λ_1 and λ_2 be defined on Γ by

$$\lambda_i = \lambda_0 - \alpha\mu_i, \quad i = 1, 2.$$

We have on Γ

$$\frac{\partial u_0}{\partial \nu} + \mu_i \Delta_\Gamma u_0 + \lambda_i u_0 = (-\lambda_0 + \alpha\mu_i + \lambda_i) u_0 = 0,$$

and

$$\mathrm{Im}(\lambda_i) = \mathrm{Im}(\lambda_0) - \mathrm{Im}(\alpha\mu_i) \geq \mathrm{Im}(\lambda_0) - |\mu_i| \max_\Gamma |\alpha| \geq 0.$$

As a result, u_0^∞ is the far field associated to the solution u_0 of the generalized impedance problem with both (λ_1, μ_1) and (λ_2, μ_2).
It should also be noted that even if μ is real (which would correspond to the thin coating case, as presented in the introduction, with $\mu = \delta$ and $\lambda = \delta k^2 n^2$), our counterexample shows that unique determination of (λ, μ) does not hold in general.
The counterexample above justifies the need to consider restricted cases where some a priori knowledge is introduced in addition to the assumption $(H0)$ given by (6). We were able to recover injectivity in the following complementary cases:

1. Assumption $(H1)$: $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$ are two constants.
   Furthermore, we assume there exist $x_0 \in \Gamma$ and $\eta > 0$ such that $\Gamma_0 = \Gamma \cap B(x_0, \eta)$ is either a portion of a plane, or a portion of a cylinder, or a portion of a sphere, and such that the set $\{x + \gamma\nu(x),\, x \in \Gamma_0,\, \gamma > 0\}$ is included in Ω.

2. Assumption $(H2)$: $\lambda \in C_I(\Gamma)$, $\mu \in \mathbb{C}$, and we assume that either of the following holds:

   - $(H2a)$ λ is fixed and known, the unknown being μ.
   - $(H2b)$ both $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\mu)$ are fixed and known, the unknown being $\mathrm{Im}(\lambda)$ and $\mathrm{Re}(\mu)$,
   - $(H2c)$ both $\mathrm{Im}(\lambda)$ and $\mathrm{Re}(\mu)$ are fixed and known, the unknown being $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\mu)$,
   - $(H2d)$ μ is fixed and known, the unknown being λ.

   Furthermore, in the three cases $(H2a)$, $(H2b)$, $(H2c)$, we assume there is no constant C such that

   $$e^{ikx\cdot d} = C(1 - 2\mathcal{T}(1)(x) - 2\mathcal{S}(\lambda)(x)) \quad \forall\, x \in \Gamma,$$

   where $\mathcal{S}$ and $\mathcal{T}$ respectively denote the single and double layer potentials on Γ (see equation (14) below). This assumption will be referred to as (HC) in the following.
   Assumption (HC) may be impossible to verify in practice, this is why we hereafter mention two particular situations where this condition is automatically satisfied (see Lemma 3.1 below):

   - the function λ is real and k^2 is not a Dirichlet eigenvalue of the operator $-\Delta$ in D,
   - the domain D is C^1 and both D and the function λ are assumed to be invariant by reflection against a plane which does not contain the direction d or by a rotation (different from identity) around an axis which is not directed by d.

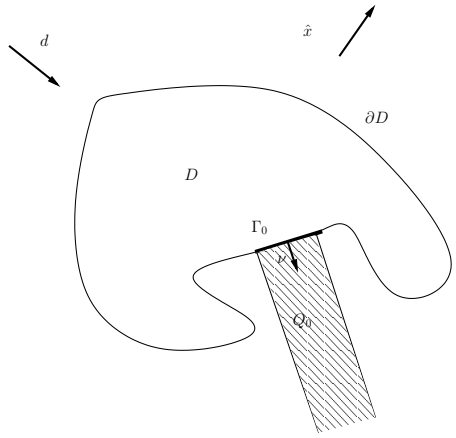


Figure 2: Illustration of assumption $(H1)$.

Let us remark that the proof under assumption $(H2d)$ follows exactly the same lines as the proof in the case $\mu = 0$ given in the previous section. We therefore shall detail the proofs only for assumptions $(H1)$, $(H2a)-(HC)$, $(H2b)-(HC)$, $(H2c)-(HC)$. The results are formulated and proved in Propositions 2, 3 and 4.

Proposition 2 *Under assumption $(H1)$ the farfield associated with one incident plane wave uniquely determines the coefficients λ and μ.*

Proof.

Using the same arguments and notation as in the first paragraph of the proof of Proposition 1, we arrive in the present case at:

$$\frac{\partial u_1}{\partial \nu} + \mu_1 \Delta_\Gamma u_1 + \lambda_1 u_1 = \frac{\partial u_1}{\partial \nu} + \mu_2 \Delta_\Gamma u_1 + \lambda_2 u_1 = 0$$

on Γ. In order to simplify the notation we set in the following $u := u_1$. If $\mu_1 = \mu_2$, then we easily conclude that $\lambda_1 = \lambda_2$ using the same arguments as in the proof of Proposition 1. If $\mu_1 \neq \mu_2$, then on Γ

$$\Delta_\Gamma u = \alpha u, \quad \frac{\partial u}{\partial \nu} = \beta u,$$

with

$$\alpha = -\frac{\lambda_2 - \lambda_1}{\mu_2 - \mu_1}, \quad \beta = -\frac{\mu_2 \lambda_1 - \mu_1 \lambda_2}{\mu_2 - \mu_1}.$$

We notice that $\alpha = -k_\Gamma^2$, where k_Γ^2 is an eigenvalue of the negative Laplace-Beltrami operator $-\Delta_\Gamma$.

Let us consider first the case when Γ contains a portion of plane $\Gamma_0 = \Gamma \cap B(x_0, \eta)$ of outward normal ν such that the set $Q_0 = \{x + \gamma\nu,\ x \in \Gamma_0,\ \gamma > 0\}$ is included in Ω. Then there exists a system of coordinates (x_1, x_2, x_2) such that $x_0(0,0,0) = 0$ and

$$Q_0 = \{x(x_1, x_2, x_3),\ \sqrt{x_1^2 + x_2^2} < \eta,\ x_3 > 0\},$$

$$\Gamma_0 = \{x(x_1, x_2, x_3),\ \sqrt{x_1^2 + x_2^2} < \eta,\ x_3 = 0\}.$$

We now consider the function $\tilde{u}$ defined in Q_0 from u by

$$\tilde{u}(x_1, x_2, x_3) = u(x_1, x_2, 0)c(x_3), \tag{7}$$

where the function c is uniquely defined by

$$\frac{d^2c}{dx_3^2} + (k^2 - k_\Gamma^2)c = 0, \quad c(0) = 1, \frac{dc}{dx_3}(0) = \beta. \tag{8}$$

In the domain Q_0 we obtain

$$\Delta\tilde{u} + k^2\tilde{u} = \Delta_\Gamma u(x_1, x_2, 0)c(x_3) + u(x_1, x_2, 0)\frac{d^2c}{dx_3^2} + k^2 u(x_1, x_2, 0)c(x_3)$$

$$= u(x_1, x_2, 0)\left(\frac{d^2c}{dx_3^2} + (k^2 - k_\Gamma^2)c\right)(x_3) = 0.$$

On the surface Γ_0 we obtain

$$\tilde{u}(x_1, x_2, 0) = u(x_1, x_2, 0), \quad \frac{\partial\tilde{u}}{\partial x_3}(x_1, x_2, 0) = \frac{\partial u}{\partial x_3}(x_1, x_2, 0).$$

The functions $\tilde{u}$ and u are both solutions of the same Helmholtz equation in Q_0, and they satisfy $\tilde{u} = u$ and $\partial_\nu\tilde{u} = \partial_\nu u$ on Γ_0. Hence, unique continuation implies $\tilde{u} = u$ in Q_0.
Since u^s satisfies the radiation condition when $||x|| \to +\infty$, we have in particular that $\lim_{x_3\to+\infty} |u^s(x_1, x_2, x_3)| = 0$. Recalling that

$$u(x_1, x_2, x_3) = u^s(x_1, x_2, x_3) + e^{ik(d_1x_1+d_2x_2+d_3x_3)},$$

we obtain

$$u(x_1, x_2, x_3) \sim e^{ik(d_1x_1+d_2x_2+d_3x_3)}, \quad x_3 \to +\infty, \tag{9}$$

and in particular when $x_1 = x_2 = 0$,

$$u(0, 0, x_3) \sim e^{ikd_3x_3}, \quad x_3 \to +\infty,$$

that is

$$c(x_3) \sim Ce^{ikd_3x_3}, \quad x_3 \to +\infty. \tag{10}$$

To see that the asymptotic behavior (10) is impossible, we have to discuss separately the three cases $k_\Gamma < k$, $k_\Gamma = k$ and $k_\Gamma > k$. In the case $k_\Gamma < k$, we obtain from (8)

$$c(x_3) = \frac{1}{2}(1 - i\frac{\beta}{\sqrt{k^2 - k_\Gamma^2}})e^{i\sqrt{k^2-k_\Gamma^2}x_3} + \frac{1}{2}(1 + i\frac{\beta}{\sqrt{k^2 - k_\Gamma^2}})e^{-i\sqrt{k^2-k_\Gamma^2}x_3}.$$

This implies that either $\sqrt{k^2 - k_\Gamma^2} = kd_3$ or $-\sqrt{k^2 - k_\Gamma^2} = kd_3$. In both cases, $c(x_3) = e^{ikd_3x_3}$ and by using again that $u(x_1, x_2, x_3) = u(x_1, x_2, 0)c(x_3)$ in Q_0, we conclude from (9) that $u(x_1, x_2, 0) = e^{ik(d_1x_1+d_2x_2)}$ in Γ_0, whence $u = e^{ikd.x}$ in Q_0, and lastly $u = e^{ikd.x}$ in Ω. Now we use $\partial u/\partial\nu = \beta u$ on Γ to see that $d.\nu$ is constant on Γ. This is forbidden by the fact there exist at least 4 points $x \in \Gamma$ such that the 4 corresponding outward normals $\nu(x)$ are different.
In the case $k_\Gamma = k$, we have

$$c(x_3) = 1 + \beta x_3,$$

while in the case $k_\Gamma > k$, we have

$$c(x_3) = \frac{1}{2}(1 + \frac{\beta}{\sqrt{k_\Gamma^2 - k^2}})e^{\sqrt{k_\Gamma^2 - k^2}x_3} + \frac{1}{2}(1 - \frac{\beta}{\sqrt{k_\Gamma^2 - k^2}})e^{-\sqrt{k_\Gamma^2 - k^2}x_3},$$

which in both cases contradicts the asymptotic behavior (10).
We now briefly consider the cases when Γ_0 is a portion of cylinder or a portion of sphere. If Γ_0 is the portion of cylinder of radius R, then in Q_0 we obtain with appropriate cylindrical coordinates (r, θ, z)

$$u(r, \theta, z) = u(R, \theta, z)c(r),$$

where the function c is uniquely defined, with $\kappa^2 := k^2 - k_\Gamma^2$, by

$$\frac{1}{r}\frac{d}{dr}(r\frac{dc}{dr}) + \kappa^2 c = 0, \quad c(R) = 1, \frac{dc}{dr}(R) = \beta.$$

Following the case $\kappa = 0$, $\kappa^2 > 0$ or $\kappa^2 = (i\tilde{\kappa})^2 < 0$, the function c is respectively a linear combination of 1 and $\log r$, a linear combination of $H_0^1(\kappa r)$ and $H_0^2(\kappa r)$ (Hankel functions of the first and second kind), or a linear combination of $I_0(\tilde{\kappa} r)$ and $K_0(\tilde{\kappa} r)$ (Modified Bessel functions). Whatever the case, the behavior of $c(r)$ when $r \to +\infty$ is not consistent with

$$u(r, \theta, z) \sim e^{ik(rd_r + zd_z)}, \quad r \to +\infty.$$

If Γ_0 is the portion of sphere of radius R, then in Q_0 we obtain with appropriate spherical coordinates (r, θ, ϕ)

$$u(r, \theta, \phi) = u(R, \theta, \phi)c(r),$$

where the function c is uniquely defined, with $\kappa^2 := k^2 - k_\Gamma^2$, by

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dc}{dr}) + \kappa^2 c = 0, \quad c(R) = 1, \frac{dc}{dr}(R) = \beta.$$

Following the case $\kappa = 0$, $\kappa^2 > 0$ or $\kappa^2 = (i\tilde{\kappa})^2 < 0$, the function c is respectively a linear combination of 1 and $1/r$, a linear combination of e^{ikr}/r and e^{-ikr}/r, or a linear combination of $e^{\tilde{k}r}/r$ and $e^{-\tilde{k}r}/r$. Whatever the case, the behavior of $c(r)$ when $r \to +\infty$ is not consistent with

$$u(r, \theta, \psi) \sim e^{ikrd_r}, \quad r \to +\infty,$$

which completes the proof. □

Proposition 3 *Under assumptions $(H2a) - (HC)$, the farfield associated with one incident plane wave uniquely determines μ.*

*Proof.*The identity

$$\frac{\partial u_1}{\partial \nu} + \mu_1 \Delta_\Gamma u_1 + \lambda u_1 = \frac{\partial u_1}{\partial \nu} + \mu_2 \Delta_\Gamma u_1 + \lambda u_1 = 0$$

on Γ leads, by denoting again $u_1 = u$ to simplify the notation, and if we assume $\mu_1 \neq \mu_2$, to

$$u = C \quad \text{and} \quad \frac{\partial u}{\partial \nu} = -C\lambda \quad \text{on} \quad \Gamma, \tag{11}$$

where $C \in \mathbb{C}$ is a constant.
We now recall the following classical representation formulas for u^s and u^i (see for example [13]).

$$u^s(x) = \mathrm{DL}(u^s)(x) - \mathrm{SL}(\frac{\partial u^s}{\partial \nu})(x), \quad \forall x \in \Omega, \tag{12}$$

$$u^i(x) = -\mathrm{DL}(u^i)(x) + \mathrm{SL}(\frac{\partial u^i}{\partial \nu})(x), \quad \forall x \in D, \tag{13}$$

where the single- and double-layer potentials SL and DL are defined for $\phi \in H^{-1/2}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$ by

$$\mathrm{SL}(\phi) = \langle \Phi(x,.), \phi(.) \rangle_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)}, \quad \mathrm{DL}(\psi) = \left\langle \frac{\partial \Phi(x,.)}{\partial \nu}, \psi(.) \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}.$$

Here $\Phi(x,y) := e^{ik||x-y||}/4\pi||x-y||$ denotes the fundamental solution of the Helmholtz equation that satisfies the Sömmerfeld radiation condition. Let γ^- (resp. γ^+) be the trace application from $H^1(D)$ (resp. V_0) into $H^{1/2}(\Gamma)$. Following [13], we define

$$\begin{aligned} &\mathcal{S} := \gamma^- \mathrm{SL} = \gamma^+ \mathrm{SL} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ &\mathcal{T} = \tfrac{1}{2}(\gamma^+ \mathrm{DL} + \gamma^- \mathrm{DL}) : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \end{aligned} \tag{14}$$

and we recall the jump relations

$$\gamma^{\pm} \mathrm{DL}\psi = \frac{1}{2}(\pm\psi + 2\mathcal{T}\psi), \quad \forall \psi \in H^{1/2}(\Gamma).$$

Then by passing to the limit $x \to \Gamma$ in (12) and (13), it follows that in the sense of trace on Γ,

$$\frac{1}{2}u^s(x) = \mathcal{T}(u^s)(x) - \mathcal{S}(\frac{\partial u^s}{\partial \nu})(x), \tag{15}$$

$$\frac{1}{2}u^i(x) = -\mathcal{T}(u^i)(x) + \mathcal{S}(\frac{\partial u^i}{\partial \nu})(x). \tag{16}$$

Using now $u = u^s + u^i$, the boundary conditions (11) and the equation (15), we obtain that

$$\frac{1}{2}u^s(x) = C\,\mathcal{T}(1)(x) - \mathcal{T}(u^i)(x) + C\mathcal{S}(\lambda)(x) + \mathcal{S}(\frac{\partial u^i}{\partial \nu})(x).$$

By subtracting (16) to the above equation we obtain

$$\frac{1}{2}(u^s(x) - u^i(x)) = C\,\mathcal{T}(1)(x) + C\mathcal{S}(\lambda)(x),$$

so that by using once again $u^s + u^i = C$ on Γ,

$$u^i(x) = \frac{C}{2}(1 - 2\mathcal{T}(1)(x) - 2\mathcal{S}(\lambda)(x)) \tag{17}$$

on Γ, which is forbidden by assumption (HC). □

Remark 2 *We point out that since $\lambda \in L^\infty(\Gamma)$, $\mathcal{S}(\lambda)$ reduces on Γ to*

$$\mathcal{S}(\lambda)(x) = \int_\Gamma \lambda(y)\Phi(x,y)\,ds(y)$$

and, if the boundary Γ is of class C^1 or is a polyhedral, $\mathcal{T}(1)$ reduces on Γ to

$$\mathcal{T}(1)(x) = \int_\Gamma \frac{\partial \Phi}{\partial \nu_y}(x,y)\,ds(y).$$

Proposition 4 *Under assumptions $(H2b)-(HC)$ or $(H2c)-(HC)$, the farfield associated with one incident plane wave uniquely determines the missing parts of λ and μ.*

*Proof.*The identity

$$\frac{\partial u_1}{\partial \nu} + \mu_1 \Delta_\Gamma u_1 + \lambda_1 u_1 = \frac{\partial u_1}{\partial \nu} + \mu_2 \Delta_\Gamma u_1 + \lambda_2 u_1 = 0$$

on Γ leads, by denoting again $u_1 = u$ to simplify the notation, and if we assume $\mu_1 \neq \mu_2$, to

$$-\Delta_\Gamma u = \frac{1}{\mu_2 - \mu_1}(\lambda_2 - \lambda_1)u.$$

By multiplying the above equation by $\overline{u}$ and by integrating by part over Γ, we obtain

$$\int_\Gamma |\nabla_\Gamma u|^2\,ds = \frac{1}{\mu_2 - \mu_1}\int_\Gamma (\lambda_2 - \lambda_1)|u|^2\,ds.$$

The assumption $(H2b)$ implies that $\lambda_2 - \lambda_1 \in i\mathbb{R}$ and $\mu_2 - \mu_1 \in \mathbb{R}$, while the assumption $(H2c)$ implies that $\lambda_2 - \lambda_1 \in \mathbb{R}$ and $\mu_2 - \mu_1 \in i\mathbb{R}$. In both cases $(\lambda_2 - \lambda_1)/(\mu_2 - \mu_1) \in i\mathbb{R}$. By taking the real part of the above equation we obtain

$$\int_\Gamma |\nabla_\Gamma u|^2\,ds = 0,$$

whence there exists a constant C such that $u = C$ on Γ. This implies that $\Delta_\Gamma u = 0$ and by using the same arguments as in the proof of Proposition 1, one obtains in particular that $\lambda_1 = \lambda_2 := \lambda$. We are then in the same configuration as in (11), and therefore one can complete the proof as for Proposition 3. □

Now we specify two particular situations where the condition (HC) is automatically satisfied.

Lemma 3.1 *Let $\lambda \in L^\infty(\Gamma)$ and d be a unitary vector of $\mathbb{R}^3$ and assume that one of the following holds:*

1. *The function λ is real and k^2 is not a Dirichlet eigenvalue of the operator $-\Delta$ in D.*
2. *The domain D is C^1 and both D and the function λ are assumed to be invariant by reflection against a plane which does not contain the direction d or by a rotation (different from identity) around an axis which is not directed by d.*

Then, there exists no constant C such that

$$e^{ikx\cdot d} = C(1 - 2\mathcal{T}(1)(x) - 2\mathcal{S}(\lambda)(x)) \quad \forall\, x \in \Gamma. \tag{18}$$

*Proof.*Let us consider first the case when λ is real and k^2 is not a Dirichlet eigenvalue of the operator $-\Delta$ in D. Taking the imaginary part of (18), we obtain on Γ

$$\mathrm{Im}\left(\frac{e^{ikx\cdot d}}{2C}\right) = -\mathrm{Im}(\mathcal{T})(1)(x) - \mathrm{Im}(\mathcal{S})(\lambda)(x). \tag{19}$$

Let us define the two functions on $D \cap \Omega$,

$$f_1(x) = \mathrm{Im}\left(\frac{e^{ikx\cdot d}}{2C}\right), \quad f_2(x) = -\mathrm{Im}(\mathrm{DL})(1)(x) - \mathrm{Im}(\mathrm{SL})(\lambda)(x).$$

Considering the regularity of

$$\mathrm{Im}(\mathrm{DL})(1)(x) = \int_\Gamma \frac{\partial .}{\partial \nu_y}(\mathrm{Im}(\Phi(x,y))\, ds(y), \quad x \in D \cap \Omega$$

and

$$\mathrm{Im}(\mathrm{SL})(\lambda)(x) = \int_\Gamma \mathrm{Im}(\Phi(x,y))\lambda(y)\, ds(y), \quad x \in D \cap \Omega,$$

since $\mathrm{Im}(\Phi(x,y)) = f(||x-y||)$ where $f(r) = \sin(kr)/4\pi r$ in a C^∞ function, it is readily shown by using differentiation under the integral sign that for general Lipschitz domain D, the function f_2 is in $C^1(\mathbb{R}^3)$. First, this implies that the traces on Γ of functions f_1 and f_2 coincide with the left-hand side and the right-hand side of equation (19), respectively. Secondly, f_1 and f_2 are functions in $C^1(\mathbb{R}^3)$ which both satisfy the Helmholtz equation in D and Ω. Hence they satisfy the Helmholtz equation in $\mathbb{R}^3$. Since their traces on Γ coincide, and since k^2 is not a Dirichlet eigenvalue of the negative laplacian in D, f_1 and f_2 coincide in D, and then in $\mathbb{R}^3$ by unique continuation. But f_2 tends to 0 when $||x|| \to +\infty$, while f_1 does not, which completes the proof for the first case.

Let us now consider the second case where D and λ are assumed to be invariant by a transformation S which is either a reflection against a plane P with $d \notin P$ or a rotation (different from identity) around an axis A with $d \notin A$. Without loss of generality, we assume that the plane P and the axis A contains O. We have

$$\mathcal{T}(1)(x) = \int_\Gamma \frac{\partial \Phi}{\partial \nu_y}(x,y)\, ds(y) = \int_\Gamma \frac{\partial \phi}{\partial \nu_y}(x-y)\, ds(y), \quad \phi(z) := \frac{e^{ik\|z\|}}{4\pi\|z\|},$$

$$\mathcal{S}(\lambda)(x) = \int_\Gamma \lambda(y)\Phi(x,y)\, ds(y) = \int_\Gamma \lambda(y)\phi(x-y)\, ds(y).$$

We use the new variables $x = Sx'$ and $y = Sy'$, and the new function ϕ' defined by $\phi(x) = \phi'(x') = \phi(Sx')$. We have for some $x \in \Gamma$,

$$\mathcal{T}(1)(x) = \int_\Gamma \nabla_y \phi(x-y).\nu_y\, ds(y) = \int_\Gamma \nabla_y \phi(S(x'-y')).\nu_y\, ds(y).$$

Using the chain rule $\nabla_{y'}\phi'(x'-y') = \nabla_y\phi(S(x'-y')).S$ and $\nu_y = S\nu_{y'}$, we obtain

$$\mathcal{T}(1)(x) = \int_\Gamma \nabla_{y'}\phi'(x'-y').\nu_{y'}\, ds(y).$$

By using the change of variable $y = Sy'$ in the integral, we obtain

$$\mathcal{T}(1)(x) = \int_\Gamma \nabla_{y'}\phi'(x'-y').\nu_{y'}\, ds(y').$$

Finally, since $\|Sx'\| = \|x'\|$ and given the form of ϕ, $\phi'(x') = \phi(x')$, and thus $\mathcal{T}(1)(x) = \mathcal{T}(1)(x')$. As a result $\mathcal{T}(1)(Sx) = \mathcal{T}(1)(x)$ for all $x \in \Gamma$. We prove the same way that $\mathcal{S}(\lambda)(Sx) = \mathcal{S}(\lambda)(x)$ for all $x \in \Gamma$, and if (18) holds for some constant C, then for all $x \in \Gamma$

$$e^{ikSx\cdot d} = e^{ikx\cdot d},$$

that is for some $n \in \mathbb{Z}$

$$Sx \cdot d = x \cdot d + \frac{2n\pi}{k}.$$

If x describes a sufficiently small part of Γ which contains the points x_0 such that $Sx_0 = x_0$, then $n = 0$. In such small part of Γ, we hence have $Sx - x \perp d$. Since d does not belong to the plane or the axis of symmetry, we conclude that $(Sx - x).d$ cannot vanish in the small part of Γ we consider, which completes the proof. □

Remark 3 *If the boundary Γ is C^1, under either of assumptions $(H2a)$, $(H2b)$, $(H2c)$, uniqueness of (λ, μ) from two far fields u_1^∞ and u_2^∞ generated by two plane waves $u_1^i = e^{ikx\cdot d_1}$ and $u_2^i = e^{ikx\cdot d_2}$ with $d_1 \neq d_2$ holds without assuming (HC). This follows from existence of two constants C_1 and C_2 such that (17) is satisfied for u_1^i and u_2^i respectively, and then differentiation of $C_2u_1^i(x) - C_1u_2^i(x) = 0$ on Γ along the tangent planes at two points x_1 and x_2 of Γ which are perpendicular to d_1 and d_2 respectively.*

4 Stability

4.1 The classical impedance problem

The analysis of stability for the inverse problem here is based on the Fréchet derivative of operator $T : L^\infty_+(\Gamma) \to L^2(S^2)$. We recall that this operator is differentiable in the sense of Fréchet if there exists a linear continuous operator $dT_\lambda : L^\infty(\Gamma) \to L^2(S^2)$ and a function $\varepsilon_\lambda : L^\infty(\Gamma) \to L^2(S^2)$ which satisfy for all $h \in L^\infty(\Gamma)$ such that $\lambda + h \in L^\infty_+(\Gamma)$,

$$T(\lambda + h) = T(\lambda) + dT_\lambda(h) + ||h||_{L^\infty(\Gamma)}\varepsilon_\lambda(h), \tag{20}$$

with $\varepsilon_\lambda(h) \to 0$ in $L^2(S^2)$ when $||h||_{L^\infty(\Gamma)} \to 0$. We have the following proposition.

Proposition 5 *The operator T is differentiable and its Fréchet derivative is the operator $dT_\lambda : L^\infty(\Gamma) \to L^2(S^2)$ which maps h to v_h^∞ such that*

$$v_h^\infty(\hat{x}) = \int_\Gamma p(y,\hat{x})u(y)h(y)\, ds(y), \quad \forall \hat{x} \in L^2(S^2),$$

where

$$p(y,\hat{x}) = \Phi^\infty(y,\hat{x}) + p^s(y,\hat{x}),$$

$p^s(.,\hat{x})$ is the solution of problem (2) in which u^i is replaced by $\Phi^\infty(.,\hat{x})$. Moreover, the operator $dT_\lambda : C_I(\Gamma) \to L^2(S^2)$ is injective.

Proof.

We denote u^s and u_h^s, the solutions of (2) which are associated to λ and $\lambda+h$ in $L_+^\infty(\Gamma)$. In order to prove differentiability, we first prove continuity. As we already remarked at the end of section 2, if we denote $K = -J_R^{-1}K_R$ and $F = J_R^{-1}F_R$ in order to simplify notations, u^s and u_h^s are respectively solutions in $H^1(\Omega_R)$ of

$$(I-K)u^s = F, \quad (I-K_h)u_h^s = F_h,$$

where K_h and F_h satisfy $K_0 = K$, $F_h = F$ and depend continuously on h. By denoting $A = I-K$ and $A_h = I-K_h$, we already know that A^{-1} and A_h^{-1} exist and are bounded. Then the family of operators A_h meet the assumptions of Theorem 10.1 in [10], and provided that $|||A^{-1}(A_h - A)||| < 1$, which happens when $||h||_{L^\infty(\Gamma)}$ is sufficiently small, we obtain the error estimate

$$||u_h^s - u^s||_{H^1(\Omega_R)} \leq \frac{|||A^{-1}|||}{1-|||A^{-1}(A_h-A)|||}\left(||(A_h-A)u^s||_{H^1(\Omega_R)} + ||F_h - F||_{H^1(\Omega_R)}\right).$$

The continuity of the operator T follows.
Now let us denote $e_h^s = u_h^s - u^s$. The function e_h^s satisfies the boundary condition on Γ

$$\frac{\partial e_h^s}{\partial \nu} + \lambda e_u^s = -h(u_h^s + u^i).$$

This leads to the fact that there exists $C > 0$ (independent of h) such that

$$||e_h^s||_{H^1(\Omega_R)} = ||u_h^s - u^s||_{H^1(\Omega_R)} \leq C||h||_{L^\infty(\Gamma)}. \tag{21}$$

Finally, let us define v_h^∞ as the far field associated to the scattered field v_h^s, where v_h^s is the solution of problem (2) with data $f = -h(u^s + u^i)$. The mapping $h \to v_h^\infty$ is clearly a linear continuous operator $L^\infty(\Gamma) \mapsto L^2(S^2)$. We obtain

$$\frac{\partial(e_h^s - v_h^s)}{\partial \nu} + \lambda(e_h^s - v_h^s) = -he_h^s.$$

It follows that $u_h^s - u^s - v_h^s$ is the solution of problem (2) with data $f = -he_h^s$. By using (21), it is clear that T is differentiable in the sense of Fréchet with $dT_\lambda(h) = v_h^\infty$. We have from (5),

$$v_h^\infty(\hat{x}) = \int_\Gamma \{v_h^s(y)\frac{\partial \Phi^\infty(y,\hat{x})}{\partial \nu(y)} - \frac{\partial v_h^s}{\partial \nu}(y)\Phi^\infty(y,\hat{x})\}\, ds(y), \quad \forall \hat{x} \in S^2.$$

$$v_h^\infty(\hat{x}) = \int_\Gamma \{v_h^s(\frac{\partial \Phi^\infty}{\partial \nu} + \lambda\Phi^\infty) + h(u^s + u^i)\Phi^\infty\}\, ds(y).$$

Using the identity

$$\int_\Gamma (\frac{\partial p^s}{\partial \nu} + \lambda p^s)v_h^s\, ds(y) = \int_\Gamma (\frac{\partial v_h^s}{\partial \nu} + \lambda v_h^s)p^s\, ds(y),$$

the definitions of p^s and v_h^s lead to

$$\int_\Gamma (\frac{\partial \Phi^\infty}{\partial \nu} + \lambda\Phi^\infty)v_h^s\, ds(y) = \int_\Gamma h(u^s + u^i)p^s\, ds(y),$$

As a result, we obtain

$$v_h^\infty(\hat{x}) = \int_\Gamma \{(u^s + u^i)(p^s + \Phi^\infty)h\}\, ds(y),$$

which is the result since $u = u^s + u^i$ and $p = p^s + \Phi^\infty$.
It remains to prove that $dT_\lambda : C_I(\Gamma) \to L^2(S^2)$ is injective. Assume that for $h \in C_I(\Gamma)$, we have

$$\int_\Gamma p(y, \hat{x}) u(y) h(y)\, ds(y) = 0, \quad \forall \hat{x} \in S^2.$$

By using lemma 4.1, there exists a sequence $\hat{x}_n$, $n \in \mathbb{N}$, with $p(., \hat{x}_n)|_\Gamma \to \overline{u|_\Gamma h}$ in $L^2(\Gamma)$. It follows that $u|_\Gamma h = 0$ on Γ, and by using the same arguments as in the proof of proposition 1, we conclude that $h = 0$. □

The proof of the above proposition requires the following density lemma.

Lemma 4.1 *The set of functions $p(., \hat{x})|_\Gamma$, when $\hat{x}$ ranges the whole unit sphere S^2, is dense in $L^2(\Gamma)$.*

*Proof.*For $f \in L^2(\Gamma)$, we hence assume that

$$\int_\Gamma p(y, \hat{x}) f(y)\, ds(y) = 0, \quad \forall \hat{x} \in S^2. \tag{22}$$

Let u^s be the solution of (2) which is associated to data f. We have

$$\int_\Gamma (p^s \frac{\partial u^s}{\partial \nu} - \frac{\partial p^s}{\partial \nu} u^s)\, ds(y) = 0,$$

which leads by using the boundary condition for u^s, i.e. $\partial u^s/\partial \nu + \lambda u^s = f$ on Γ,

$$\int_\Gamma (\frac{\partial p^s}{\partial \nu} + \lambda p^s) u^s\, ds(y) = \int_\Gamma f p^s\, ds(y).$$

Using now the boundary condition for p^s, i.e. $\partial p^s/\partial \nu + \lambda p^s = -(\partial \Phi^\infty/\partial \nu + \lambda \Phi^\infty)$ on Γ,

$$-\int_\Gamma (\frac{\partial \Phi^\infty}{\partial \nu} + \lambda \Phi^\infty) u^s\, ds(y) = \int_\Gamma f p^s\, ds(y).$$

From (22) we have

$$\int_\Gamma f p^s\, ds(y) = -\int_\Gamma f \Phi^\infty\, ds(y).$$

By using the two previous equalities and once again the boundary condition for u^s, it follows that

$$\int_\Gamma (u^s \frac{\partial \Phi^\infty}{\partial \nu} - \frac{\partial u^s}{\partial \nu} \Phi^\infty)\, ds(y) = 0,$$

which is exactly $u^\infty(\hat{x}) = 0$, for all $\hat{x} \in S^2$. We conclude from Rellich's lemma and unique continuation that $u^s = 0$, and hence $f = \partial u^s/\partial \nu + \lambda u^s = 0$. □

We now introduce $D_I(\Gamma)$ a finite dimensional subspace of $C_I(\Gamma)$. We have the following result of local Lipschitz stability in $D_I(\Gamma) \cap L^\infty_+(\Gamma)$.

Lemma 4.2 *For each $\lambda \in D_I(\Gamma) \cap L^\infty_+(\Gamma)$ there exist $\eta(\lambda) > 0$ and $C(\lambda) > 0$ such that for all $h \in L^\infty(\Gamma)$ verifying $\lambda + h \in D_I(\Gamma) \cap L^\infty_+(\Gamma)$ and $||h||_{L^\infty(\Gamma)} \leq \eta(\lambda)$,*

$$||h||_{L^\infty(\Gamma)} \leq C(\lambda)||u_h^\infty - u^\infty||_{L^2(S^2)},$$

where $u^\infty = T(\lambda)$ and $u_h^\infty = T(\lambda + h)$.

*Proof.*We denote $v_h^\infty = dT_\lambda(h)$ and $w_h^\infty = ||h||_{L^\infty(\Gamma)}\varepsilon_\lambda(h)$ in (20), so that

$$u_h^\infty - u^\infty = v_h^\infty + w_h^\infty.$$

From proposition 5, we deduce that $dT_\lambda : D_I(\Gamma) \to R(D_I(\Gamma))$ is injective, whence it is of continuous inverse since $D_I(\Gamma)$ is finite dimensional. Precisely, for all $\lambda \in D_I(\Gamma) \cap L^\infty_+(\Gamma)$ there exists $c(\lambda) > 0$ such that for all $h \in L^\infty(\Gamma)$ with $\lambda + h \in D_I(\Gamma) \cap L^\infty_+(\Gamma)$,

$$||h||_{L^\infty(\Gamma)} \leq c(\lambda)||v_h^\infty||_{L^2(S^2)}.$$

Setting $\varepsilon = ||u_h^\infty - u^\infty||_{L^2(S^2)}$,

$$||v_h^\infty||_{L^2(S^2)} \leq ||v_h^\infty + w_h^\infty||_{L^2(S^2)} + ||w_h^\infty||_{L^2(S^2)} = \varepsilon + ||h||_{L^\infty(\Gamma)}||\varepsilon_\lambda(h)||_{L^2(S^2)}.$$

From the two above estimates we conclude that

$$\left(\frac{1}{c(\lambda)} - ||\varepsilon_\lambda(h)||_{L^2(S^2)}\right)||h||_{L^\infty(\Gamma)} \leq \varepsilon.$$

The fact that $||\varepsilon_\lambda(h)||_{L^2(S^2)} \to 0$ when $||h||_{L^\infty(\Gamma)} \to 0$ completes the proof. □

Lastly, $K_I(\Gamma)$ denotes a compact subset of $D_I(\Gamma) \cap L^\infty_+(\Gamma)$. We have the following result of stability in $K_I(\Gamma)$, which is Lipschitz stability for the considered inverse impedance problem.

Theorem 4.3 *For all $\lambda \in K_I(\Gamma)$, there exists a positive constant $C(\lambda)$ such that for all $\tilde{\lambda} \in K_I(\Gamma)$,*

$$||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} \leq C(\lambda)\,||u^\infty - \tilde{u}^\infty||_{L^2(S^2)},$$

where $u^\infty = T(\lambda)$ and $\tilde{u}^\infty = T(\tilde{\lambda})$.

*Proof.*For $\lambda \in K_I(\Gamma)$, we have for all $\tilde{\lambda} \in K_I(\Gamma)$ such that $||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} \leq \eta(\lambda)$,

$$||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} \leq C(\lambda)\,||u^\infty - \tilde{u}^\infty||_{L^2(S^2)}, \tag{23}$$

where $\eta(\lambda)$ and $C(\lambda)$ are defined as in Lemma 4.2.
It remains to prove that (23) is still valid for $||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} > \eta(\lambda)$, perhaps with another constant $C(\lambda)$. Assume that for all $n \in \mathbb{N}$, there exists $\tilde{\lambda}_n$ such that $||\lambda - \tilde{\lambda}_n||_{L^\infty(\Gamma)} > \eta$ and

$$||\lambda - \tilde{\lambda}_n||_{L^\infty(\Gamma)} \geq n\,||u^\infty - \tilde{u}_n^\infty||_{L^2(S^2)},$$

where $T(\tilde{\lambda}_n) := \tilde{u}_n^\infty$. It follows that

$$||u^\infty - \tilde{u}_n^\infty||_{L^2(S^2)} \leq \frac{2M}{n},$$

where $M > 0$ is such that $K_I(\Gamma) \subset B(0, M)$ in $L^\infty(\Gamma)$. Since the sequence $(\tilde{\lambda}_n)$ belongs to the compact set $K_I(\Gamma)$, it follows that there exists a sub-sequence, still denoted $(\tilde{\lambda}_n)$, which converges to $\tilde{\lambda}$. By continuity of operator T, it follows that $\tilde{u}_n^\infty$ converges to $T(\tilde{\lambda}) := \tilde{u}^\infty$. From the previous inequality it follows that $u^\infty = \tilde{u}^\infty$, and from the injectivity of the restriction of T to $C_I(\Gamma)$, it follows that $\lambda = \tilde{\lambda}$, which is in contradiction with $||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} > \eta$. The proof is complete. □

Remark 4 *Let us remark that similar Lipschitz stability results were already established in [17] (see Theorem 2.4) for the Laplace equation and standard impedance problem ($\mu = 0$) with a piecewise-constant impedance. In [17], the Lipschitz constant is proved to be independent of λ, with the help of a more complex technique based on a quantification of the unique continuation principle. One can also find in [17] a quantification of the exponential blowing up of the Lipschitz stability constant with respect to the space dimension of the parameters. For other results related to the impedance problem for the Laplace equation one can refer to [4, 5].*

4.2 The generalized impedance problem

We analyze the particular case of assumption $(H2b) - (HC)$ or $(H2c) - (HC)$, the other cases can be studied similarly. The analysis of stability for the inverse problem here is based on the Fréchet derivative of operator $T : V(\Gamma) \subset L^\infty(\Gamma) \times \mathbb{C} \to L^2(S^2)$, where $V(\Gamma)$ is defined by assumptions $(H0) - (H2b) - (HC)$ or $(H0) - (H2c) - (HC)$ (that guarantee well posedness of the forward scattering problem as well as uniqueness for the inverse problem). We have the following proposition.

Proposition 6 *The operator T is differentiable in $V(\Gamma)$ and its Fréchet derivative is the operator $dT_{\lambda,\mu} : L^\infty(\Gamma) \times \mathbb{C} \to L^2(S^2)$ which maps (h, l) to v_h^∞ such that*

$$v_h^\infty(\hat{x}) = \langle p(., \hat{x}), l\Delta_\Gamma u + u\, h \rangle_{H^1(\Gamma), H^{-1}(\Gamma)}, \quad \forall \hat{x} \in L^2(S^2),$$

where

$$p(., \hat{x}) = \Phi^\infty(., \hat{x}) + p^s(., \hat{x}),$$

$p^s(., \hat{x})$ is the solution of problem (2) in which u^i is replaced by $\Phi^\infty(., \hat{x})$.
Moreover, the operator $dT_{\lambda,\mu} : C_I(\Gamma) \times \mathbb{C} \to L^2(S^2)$ is injective under assumption $(H2b) - (HC)$ or $(H2c) - (HC)$.

*Proof.*By reasoning exactly as in the proof of proposition 5, we obtain that T is differentiable and that $dT_{\lambda,\mu}(h, l)$ coincide with the far field $v_{h,l}^\infty$ associated to the scattered field $v_{h,l}^s$, where $v_{h,l}^s$ is the solution of problem (2) with data $f = -l\Delta_\Gamma(u^s + u^i) - h(u^s + u^i) = -l\Delta_\Gamma u - hu$. From (5), we have for all $\hat{x} \in S^2$

$$v_{h,l}^\infty(\hat{x}) = \left\langle v_{h,l}^s, \frac{\partial \Phi^\infty(., \hat{x})}{\partial \nu(y)} \right\rangle_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} - \left\langle \frac{\partial v_{h,l}^s}{\partial \nu}(y), \Phi^\infty(., \hat{x}) \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}.$$

Since on Γ

$$\frac{\partial v_{h,l}^s}{\partial \nu} + \mu\Delta_\Gamma v_{h,l}^s + \lambda v_{h,l}^s = -l\Delta_\Gamma u - hu,$$

we obtain that

$$v_{h,l}^{\infty}(\hat{x}) = \left\langle v_{h,l}^{s}, \frac{\partial \Phi^{\infty}(.,\hat{x})}{\partial \nu} \right\rangle_{H^1(\Gamma),H^{-1}(\Gamma)}$$
$$+ \left\langle \mu \Delta_{\Gamma} v_{h,l}^{s} + \lambda v_{h,l}^{s} + l \Delta_{\Gamma} u + hu, \Phi^{\infty} \right\rangle_{H^{-1}(\Gamma),H^1(\Gamma)}.$$

By using the fact that

$$\left\langle \Delta_{\Gamma} v_{h,l}^{s}, \Phi^{\infty} \right\rangle_{H^{-1}(\Gamma),H^1(\Gamma)} = -\int_{\Gamma} \nabla v_{h,l}^{s}.\nabla \Phi^{\infty}\, ds = \left\langle v_{h,l}^{s}, \Delta_{\Gamma} \Phi^{\infty} \right\rangle_{H^1(\Gamma),H^{-1}(\Gamma)},$$

$$v_{h,l}^{\infty}(\hat{x}) = \left\langle v_{h,l}^{s}, \frac{\partial \Phi^{\infty}}{\partial \nu} + \mu \Delta_{\Gamma} \Phi^{\infty} + \lambda \Phi^{\infty} \right\rangle_{H^1(\Gamma),H^{-1}(\Gamma)} + \langle \Phi^{\infty}, l \Delta_{\Gamma} u + hu \rangle_{H^1(\Gamma),H^{-1}(\Gamma)}$$

Using now the definition of p^s, we have on Γ

$$\frac{\partial p^s}{\partial \nu} + \mu \Delta_{\Gamma} p^s + \lambda p^s = -(\frac{\partial \Phi^{\infty}}{\partial \nu} + \mu \Delta_{\Gamma} \Phi^{\infty} + \lambda \Phi^{\infty}).$$

Thus

$$\left\langle v_{h,l}^{s}, \frac{\partial \Phi^{\infty}}{\partial \nu} + \mu \Delta_{\Gamma} \Phi^{\infty} + \lambda \Phi^{\infty} \right\rangle_{H^1(\Gamma),H^{-1}(\Gamma)} = -\left\langle v_{h,l}^{s}, \frac{\partial p^s}{\partial \nu} + \mu \Delta_{\Gamma} p^s + \lambda p^s \right\rangle_{H^1(\Gamma),H^{-1}(\Gamma)}$$

$$= -\left\langle p^s, \frac{\partial v_{h,l}^{s}}{\partial \nu} + \mu \Delta_{\Gamma} v_{h,l}^{s} + \lambda v_{h,l}^{s} \right\rangle_{H^1(\Gamma),H^{-1}(\Gamma)} = \langle p^s, l \Delta_{\Gamma} u + hu \rangle_{H^1(\Gamma),H^{-1}(\Gamma)}.$$

Finally we obtain

$$v_{h,l}^{\infty}(\hat{x}) = \langle p, l \Delta_{\Gamma} u + hu \rangle_{H^1(\Gamma),H^{-1}(\Gamma)},$$

which is the desired result.
It follows from lemma 4.4 that $l\Delta_{\Gamma} u + hu = 0$ on Γ, and by using the same arguments as in the proof of proposition 4, we conclude that $h = 0$ and $l = 0$.
□

The proof of the above proposition requires the following lemma.

Lemma 4.4 *If $f \in H^{-1}(\Gamma)$ satisfies*

$$\langle p(.,\hat{x}), f \rangle_{H^1(\Gamma),H^{-1}(\Gamma)} = 0, \quad \forall \hat{x} \in S^2, \tag{24}$$

then $f = 0$.

*Proof.*Let u^s be the solution of (2) which is associated to data f. We have

$$\left\langle p^s, \frac{\partial u^s}{\partial \nu} \right\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)} - \left\langle \frac{\partial p^s}{\partial \nu}, u^s \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0,$$

which leads by using the boundary condition for u^s, i.e. $\partial u^s/\partial \nu + \Delta_{\Gamma} u^s + \lambda u^s = f$ on Γ,

$$\left\langle u^s, \frac{\partial p^s}{\partial \nu} + \mu \Delta_{\Gamma} p^s + \lambda p^s \right\rangle_{H^1(\Gamma),H^{-1}(\Gamma)} = \langle p^s, f \rangle_{H^1(\Gamma),H^{-1}(\Gamma)}.$$

Using now the boundary condition for p^s, i.e. $\partial p^s/\partial\nu+\mu\Delta_\Gamma p^s+\lambda p^s = -(\partial\Phi^\infty/\partial\nu+\mu\Delta_\Gamma\Phi^\infty+\lambda\Phi^\infty)$ on Γ,

$$-\left\langle u^s, \frac{\partial\Phi^\infty}{\partial\nu}+\mu\Delta_\Gamma\Phi^\infty+\lambda\Phi^\infty\right\rangle_{H^1(\Gamma),H^{-1}(\Gamma)} = \langle p^s, f\rangle_{H^1(\Gamma),H^{-1}(\Gamma)}\,.$$

From (24) we have

$$\langle p^s, f\rangle_{H^1(\Gamma),H^{-1}(\Gamma)} = -\langle \Phi^\infty, f\rangle_{H^1(\Gamma),H^{-1}(\Gamma)}\,.$$

By using the two previous equalities and once again the boundary condition for u^s, it follows that

$$\left\langle u^s, \frac{\partial\Phi^\infty}{\partial\nu}\right\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)} - \left\langle \frac{\partial u^s}{\partial\nu}, \Phi^\infty\right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0,$$

which is exactly $u^\infty(\hat{x}) = 0$, for all $\hat{x}\in S^2$. We conclude from Rellich's lemma and unique continuation that $u^s = 0$, and hence $f = \partial u^s/\partial\nu+\Delta_\Gamma u^s+\lambda u^s = 0$. □

By using the same arguments as for the classical impedance problem, we obtain the following Lipschitz stability for the inverse problem when the parameters are restricted to the subset $K(\Gamma)$ defined as follows. Let $D(\Gamma)$ be a finite dimensional subspace of $C_I(\Gamma)\times\mathbb{C}$, $K(\Gamma)$ a compact subset of $D(\Gamma)\cap V(\Gamma)$.

Theorem 4.5 *For all $(\lambda,\mu)\in K(\Gamma)$, there exists a positive constant $C(\lambda,\mu)$ such that for all $(\tilde\lambda,\tilde\mu)\in K(\Gamma)$,*

$$||\lambda-\tilde\lambda||_{L^\infty(\Gamma)}+|\mu-\tilde\mu| \le C(\lambda,\mu)\,||u^\infty-\tilde u^\infty||_{L^2(S^2)},$$

where $u^\infty = T(\lambda,\mu)$ and $\tilde u^\infty = T(\tilde\lambda,\tilde\mu)$.

5 The backscattering inverse problem

Extending the results of previous sections to the case of backscattering data would be an interesting issue. However, this task seems to be challenging, especially for the global uniqueness question (to the best of our knowledge, this is still an open problem). We shall restrict ourselves in the following preliminary investigations to the local Lipschitz stability questions and only consider the classical impedance problem ($\mu = 0$).

We shall use a notation that explicitly indicates the dependence of the solution of problem (2) with respect to the direction d of the incident wave u^i. Consequently, from now on, the scattered field is denoted by $u^s(.,d)$, the corresponding farfield is denoted by $u^\infty(.,d)$ and the total field is denoted by $u(.,d)$.

The inverse impedance problem from backscattering data correspond to the reconstruction of the impedance λ from the values of $\tilde u^\infty(d) = u^\infty(-d,d)$, for all $d\in S^2$. Hence we shall consider the (non linear) operator $\tilde T:\lambda\to\tilde u^\infty$ and study the injectivity of the corresponding Fréchet derivative.

Proposition 7 *The operator $\tilde T$ is differentiable and its Fréchet derivative is the operator $d\tilde T_\lambda : L^\infty(\Gamma)\to L^2(S^2)$ which maps h to z_h^∞ such that*

$$z_h^\infty(d) = \frac{1}{4\pi}\int_\Gamma u^2(y,d)h(y)\,ds(y), \quad \forall d\in L^2(S^2).$$

Moreover, the operator $d\tilde{T}_\lambda : C_I(\Gamma) \to L^2(S^2)$ *is injective.*

*Proof.*By following exactly the proof of proposition 5, it follows that $\tilde{T}$ is differentiable and its Fréchet derivative is the operator $d\tilde{T}_\lambda : L^\infty(\Gamma) \to L^2(S^2)$, with $z_h^\infty = d\tilde{T}_\lambda(h)$ such that

$$z_h^\infty(d) = \int_\Gamma p(y,-d)u(y,d)h(y)\,ds(y), \quad \forall d \in L^2(S^2),$$

where $p(.,\hat{x})$ and $u(.,d)$ are the total fields due to the incident waves $\frac{1}{4\pi}e^{-ik\hat{x}.y}$ and $e^{ikd.y}$, respectively. As a result, $p(.,-d) = \frac{1}{4\pi}u(.,d)$ in Ω, which completes the first part of the proposition.
Assume that $d\tilde{T}_\lambda(h) = 0$ for some $h \in C_I(\Gamma)$. Let us prove that $h = 0$. If not, there exists $x_0 \in \Gamma_i$, for some $i = 1, ..., I$, with $h(x_0) \neq 0$, whence for example $\mathrm{Re}(h(x_0)) > 0$. By continuity of h on Γ_i, there exists $\eta > 0$ such that $\mathrm{Re}(h(x)) > 0$ for $x \in \Gamma_0 := B(x_0,\eta) \cap \Gamma$. By using once again the lemma 4.1, there exists a sequence $d_n \in S^2$ such that $u(.,d_n)|_\Gamma \to \chi_{\Gamma_0}$ in $L^2(\Gamma)$ when $n \to +\infty$, where χ denotes the indicator function. By using Cauchy-Schwarz inequality, we have

$$\left|\int_\Gamma (u^2(y,d_n) - \chi_{\Gamma_0}(y))h(y)\,ds(y)\right|$$
$$\leq ||u(.,d_n) - \chi_{\Gamma_0}||_{L^2(\Gamma)}||u(.,d_n) + \chi_{\Gamma_0}||_{L^2(\Gamma)}||h||_{L^\infty(\Gamma)}.$$

Passing to the limit $n \to +\infty$, it follows that

$$\int_{\Gamma_0} \mathrm{Re}(h(y))\,ds(y) = 0,$$

and hence $\mathrm{Re}(h) = 0$ for all $x \in \Gamma_0$, which leads to a contradiction. □

It follows from proposition 7 that we have the same result of local Lipschitz stability in $D_I(\Gamma) \cap L^\infty_+(\Gamma)$ as in lemma 4.2.

Proposition 8 *For each* $\lambda \in D_I(\Gamma) \cap L^\infty_+(\Gamma)$ *there exist* $\eta(\lambda) > 0$ *and* $C(\lambda) > 0$ *such that for all* $h \in L^\infty(\Gamma)$ *verifying* $\lambda + h \in D_I(\Gamma) \cap L^\infty_+(\Gamma)$ *and* $||h||_{L^\infty(\Gamma)} \leq \eta(\lambda)$*,*

$$||h||_{L^\infty(\Gamma)} \leq C(\lambda)||z_h^\infty - z^\infty||_{L^2(S^2)},$$

where $z^\infty = \tilde{T}(\lambda)$ *and* $z_h^\infty = \tilde{T}(\lambda + h)$*.*

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Centre de recherche INRIA Saclay – Île-de-France
Parc Orsay Université - ZAC des Vignes
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